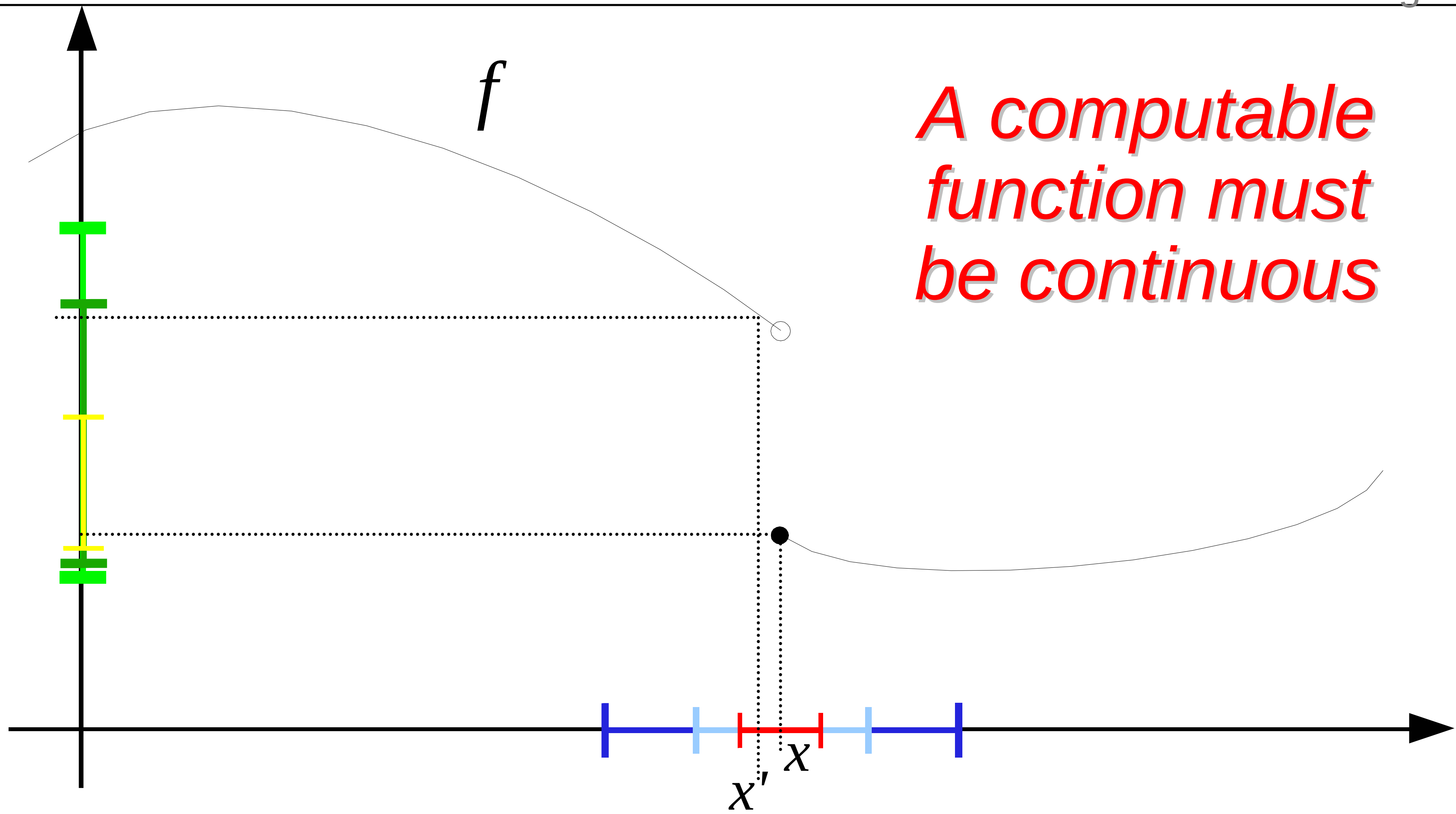
Uniformly Computable Real Functions

TECHNISCHE UNIVERSITÄT DARMSTADT
Martin Ziegler



A computable function must be continuous

$x \in \mathbb{R}$ computable $\Leftrightarrow$ $|x - a_n/2^{n+1}| \leq 2^{-n}$ for recursive $(a_n) \subseteq \mathbb{Z}$

Computable Weierstrass Theorem

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Theorem: For $f:[0,1]\to\mathbb{R}$ the following are equivalent:

a) There is an algorithm converting any seq. $q_n\in\mathbb{D}_{n+1}$ with $|x-q_n|\leq 2^{-n}$ into $p_m\in\mathbb{D}_{m+1}$ with $|f(x)-p_m|\leq 2^{-m}$

b) There is an algorithm printing a sequence (of degrees and coefficient lists of) $(P_n)\subseteq\mathbb{D}[X]$ with $\|f-P_n\|\leq 2^{-n}$

c) The real sequence $f(q)$, $q\in\mathbb{D}\cap[0,1]$, is computable & f admits a computable modulus of uniform continuity

$|x-y|\leq 2^{-\mu(m)} \Rightarrow |f(x)-f(y)|\leq 2^{-m}$

Proof: a) $\Rightarrow$ c) $\Rightarrow$ b)

Call $(r_m)\subseteq\mathbb{R}$ computable iff an algorithm can print, on input $n,m\in\mathbb{N}$, some $q\in\mathbb{D}_{n+1}$ with $|r_m-q|\leq 2^{-n}$.
$\mathbb{D} := \bigcup_n \mathbb{D}_n$, $\quad\mathbb{D}_n := \{\, a/2^n : a\in\mathbb{Z}\,\}$

Exercises: Computable Real Functions

a) f computable $\Rightarrow$ same for any restriction

b) $\exp, \sin, \cos, \ln(1+x)$ are computable functions

c) For a computable sequence $\underline{a}=(a_n)$,
the power series $x \to \sum_n a_n \cdot x^n$ is computable
on $(-r,r)$ for $r < R(\underline{a}) := 1/\limsup_n |a_n|^{1/n}$

d) Let $f \in C[0,1]$ be computable. Then so are
$\int f: x \to \int_0^x f(t)\,dt$ and $\max(f): x \to \max\{f(t): t \leq x\}$.

e) If $(x,m) \to f_m(x)$ computable with $|f_n - f_m|_\infty \leq 2^{-n}+2^{-m}$
then $\lim_n f_n$ is computable.

uncomputable in general

f) For computable $a \in \mathbb{R}$, $f:[0,a] \to \mathbb{R}$, and

To compute $f:\mathbb{R} \to \mathbb{R}$: convert any sequence $q_n \in \mathbb{D}_{n+1}$
with $|x-q_n| \leq 2^{-n}$ into $p_m \in \mathbb{D}_{m+1}$ with $|f(x)-p_m| \leq 2^{-m}$