Computable Real Numbers

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Theorem: For $r\in\mathbb{R}$, the following are equivalent:

Call $r\in\mathbb{R}$ computable if

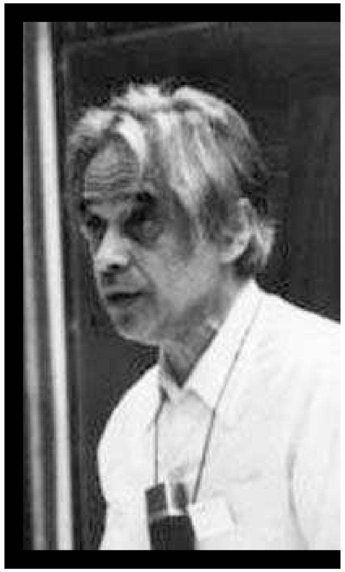
There is an algorithm which, given $n\in\mathbb{N}$, prints $b_n\in\{0,1\}$ where $r=\sum_n b_n 2^{-n}$

a) r has a computable binary expansion

b) There is an algorithm printing, on input $n\in\mathbb{N}$, some $a\in\mathbb{Z}$ with $|r-a/2^{n+1}|\leq 2^{-n}$.

$\Leftrightarrow r\in [q_n\pm\varepsilon_n]$

c) There is an algorithm printing two sequences $(q_n)\subseteq\mathbb{Q}$ and (ε_n) with $|r-q_n|\leq\varepsilon_n\to 0$



numerators+ denominators

b) $\Leftrightarrow$ c) holds *uniformly*, $\Leftrightarrow$ a) does not [Turing'37]

interval arithmetic

Ernst Specker (1949): (c) $\Leftrightarrow$ *Halting problem* plus (d)

d) There is an algorithm printing $(q_n)\subseteq\mathbb{Q}$ with $q_n\to r$.

$H := \{ \langle B,\underline{x}\rangle : \text{algorithm } B \text{ terminates on input } \underline{x} \} \subseteq \mathbb{N}$

Exercises: Computable Reals

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a) Every rational has a computable binary expansion

b) Every dyadic rational has two binary expansions

c) Computable binary expansion $\Leftrightarrow$ computable real

d) If a,b are computable, then also $a+b$, $a\cdot b$, $1/a$ $(a\neq 0)$

e) Fix $p\in\mathbb{R}[X]$. Then p's coefficients are computable
$\Leftrightarrow$ $p(x)$ is computable for all computable x.

f) The degree of every $p\in\mathbb{R}[X]$ is computable.

g) Every algebraic number is computable; and so is π.

h) If x is computable, then so are $\exp(x)$, $\sin(x)$, $\log(x)$

j) For every computable x, $\mathrm{sign}(x)$ is computable.

k) Specker's sequence $(\sum_{k>n\in H} 2^{-n})_k$ is computable,
its limit is uncomputable, yet naively computable.

$r\in\mathbb{R}$ computable iff an algorithm can print, on input $n\in\mathbb{N}$, some $d\in\mathbb{Z}$ with $|r-d/2^{n+1}|\leq 2^{-n}$

Uniformity, Sequences and Equality Testing

Reminder: For $r\in\mathbb{R}$, the following are equivalent:

a) ∃algorithm deciding r's bin. exp

b) ∃algorithm printing on input n some $a\in\mathbb{Z}$ with $|r-a/2^{n+1}|\le 2^{-n}$.

c) ∃algorithm printing $(q_n),(\varepsilon_n)\subseteq\mathbb{Q}$ with $|r-q_n|\le\varepsilon_n\to 0$

a)⇒b)⇔c) computable transformation on algorithms
b)⇒a) 'undecidable' case split on $r\in\mathbb{Q}$

Call $(r_m)\subseteq\mathbb{R}$ computable iff an algorithm can print, on input $\langle n,m\rangle\in\mathbb{N}$, some $a\in\mathbb{Z}$ with $|r_m-a/2^{n+1}|\le 2^{-n}$.

In numerics, don't test for (in-)equality!

Fact: There exists a computable sequence $(r_m)\subseteq[0,1]$ such that $\{\, m : r_m\neq 0 \,\}$ is the Halting problem H.

$H := \{\, \langle B,\underline{x}\rangle : \text{algorithm } B \text{ terminates on input } \underline{x} \,\} \subseteq \mathbb{N}$