TECHNISCHE UNIVERSITÄT DARMSTADT

Mathematik

Computability and Complexity in Analysis

IRTG 1529

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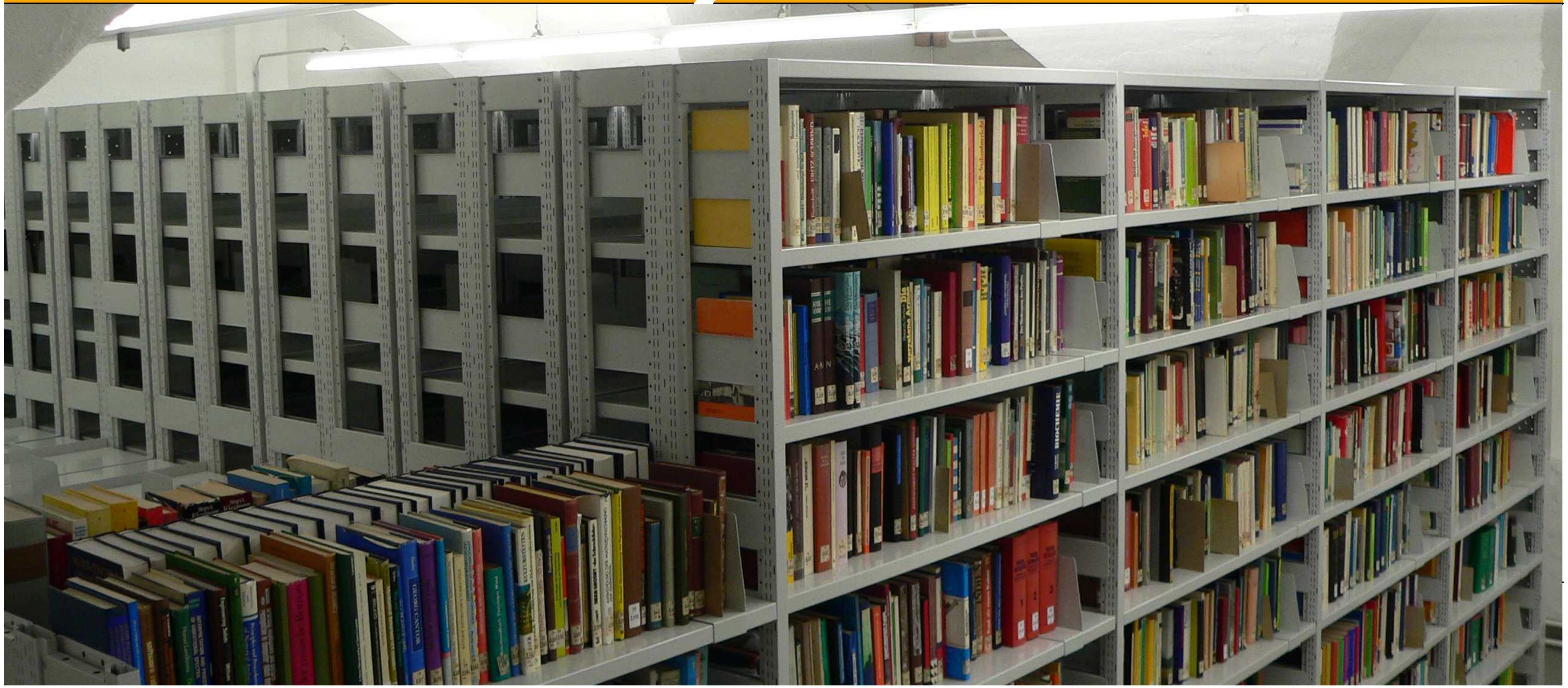
Martin Ziegler



Contents

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- in-/computability,
- Halting problem,
- Reduction, enumerability
- computability of real numbers
- Specker sequence,
  in-/effective convergence

Real function computability
- nonuniform vs. uniform
- *Main Theorem*
- modulus of continuity
- Computable Weierstrass
- power series
- computable join, max, $\int$
- uncomputable:
  argmax, roots, ∂_x
- Wave Equation

Minicourse Discrete Complexity
- bit-model of computation
- asymptotic runtime/memory
- example algorithms: Sieve, Euler Circuit, Edge Cover
- SAT, 3SAT, Vertex Cover, Hamilton Ciruit, TSP
- polynomial reduction
- $4SAT \le 3SAT \le$ Vertex Cover
- $\mathcal{NP}$-completeness

Real function complexity
- polyn. continuity, $1/\ln(e/x)$
- complexity of max
- Laplace/Poisson Eq.
- iRRAM

Alan M. Turing 1936

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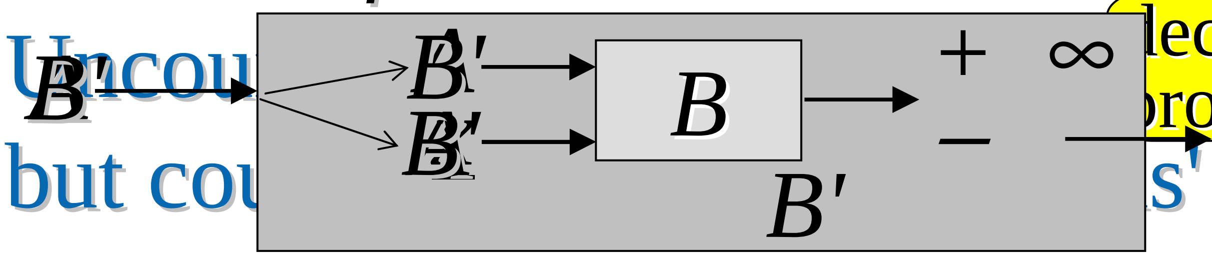
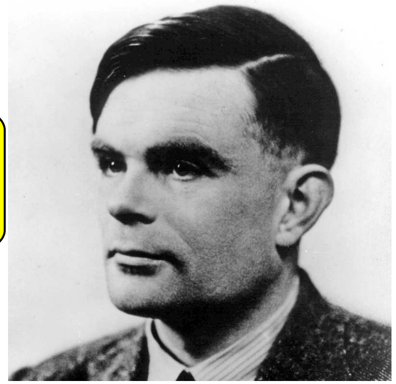
- first scientific calculations on digital computers
- *What are its fundamental limitations?*
  - Uncou…
  - but cou… s'





- Undecidable <u>Halting Problem *H*</u>: **No** algorithm *B* can always correctly ans… ~~simulator/interpreter *B*~~ ?

Given ⟨A,x⟩, does algorithm A terminate on input x?

Proof (by contradiction): Consider algo. *B'* that, on input *A*, executes *B* on ⟨*A*,*A*⟩ and, upon a positive answer, loops infinitely. How does *B'* behave on *B'* ?

- Logicians Tarski, Alonzo Church (PhD advisor)
- Kurt Gödel (1931): There exist arithmetical statements which are true but cannot be proven so.

Formalities & Tools

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e.g. "Turing machine"

'Definition:' Algorithm A decides set $L \subseteq \{0,1\}^*$ if

- on inputs $\underline{x} \in L$ prints 1 and terminates,
- on inputs $\underline{x} \notin L$ prints 0 and terminates.

all finite binary sequences

A semi-decides if terminates on $\underline{x} \in L$, else diverge.

Hilbert Hotel | Halting Problem H only *semi*-decidable

countable! | eg. $\mathcal{U}$={ algorithms } × { inputs }

Universes $\mathcal{U}$ other than $\{0,1\}^*$ (e.g. $\mathbb{N}$): encode.

Techniques: a) simulation b) diagonalization
c) dovetailing d) reduction (in/output translation)

Theorem: L decidable iff both L, L^C semi-decidable

Infinite $L \subseteq \{0,1\}^*$ is semi-decidable iff L=range(f)
for some computable injective $f:\mathbb{N} \to \{0,1\}^*$

Some Undecidable Problems

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'Definition:' Algorithm A decides set $L \subseteq \{0,1\}^*$ if

- on inputs $\underline{x} \in L$ prints 1 and terminates,
- on inputs $\underline{x} \notin L$ prints 0 and terminates.

Halting problem: $H = \{ \langle A,\underline{x}\rangle : A$ terminates on $\underline{x} \}$

Hilbert's 10th: The following set is undecidable:
$\{ \langle p\rangle \mid p \in \mathbb{N}[X_1,\ldots X_n],\ n \in \mathbb{N},\ \exists x_1\ldots x_n \in \mathbb{N}\ \ p(x_1,\ldots x_n)=0 \}$

Word Problem for finitely presented groups

Universes U other than $\{0,1\}^*$ (e.g. $\mathbb{N}$): encode.

Mortality Problem for two 21×21 matrices

Techniques: a) simulation b) diagonalization

Homeomorphy of 2 finite simplicial complexes

c) dovetailing d) reduction (in/output translation)

For $L,L' \subseteq \{0,1\}^*$ write $L \preccurlyeq L'$ if there is a computable $f: \{0,1\}^* \to \{0,1\}^*$ such that $\forall \underline{x}:\ \underline{x} \in L \iff f(\underline{x}) \in L'$.

a) L' decidable $\Rightarrow$ so L.   b) $L \preccurlyeq L' \preccurlyeq L'' \Rightarrow L \preccurlyeq L''$

Exercise Questions

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Which of the following are un-/semi-/decidable?

a) Given an integer, is it a prime number?

b) Given a finite string over +,×,(,),0,1,$X_1,\ldots X_n$ is it syntactically correct?

c) Given a Boolean formula $\varphi(X_1,\ldots X_n)$, does it have a *satisfying assignment*?

d) Given $M\in\mathbb{Z}^{n\times n}$ and $b\in\mathbb{Z}^n$, does there exists a real vector $\underline{x}$ s.t. $\underline{M}\cdot\underline{x}\leq\underline{b}$?

e) Given an algorithm *A*, input $\underline{x}$, and integer *N*, does *A* terminate on input $\underline{x}$ within *N* steps ?

f) Does a given algorithm terminate on all inputs?

g) Does given algorithm terminate on some input?