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Computable Analysis

SS 2013, Exercise Sheet #10

EXERCISE 18:

a) Let P denote a second-order polynomial in $n \in \mathbb{N}$ and in $\lambda \in \mathbb{N}^{\mathbb{N}}$.
Prove by structural induction on P that, for $\lambda \in \mathbb{N}[N]$,

   i) $P(n,\lambda)$ is a polynomial in n
   ii) whose degree depends only on $\deg(\lambda)$ and
   iii) that $\mathbb{N} \ni d \mapsto \deg P(n,N^d)$ is a polynomial in d.

b) Theorem 4.7h) in the script asserts that the closed image of a closed $\psi_<^d$–computable set under a computable function is again $\psi_<^d$–computable. We also know from Exercise 13c) that intersection of non-empty closed sets is $(\psi_>^d \times \psi_>^d, \psi_>^d)$–computable.

   i) Prove that the image of compact subsets under a computable function is $\psi_>^d$–computable. More precisely use Theorem 4.7j+k) to establish that $C(\mathbb{R}^d,\mathbb{R}^k) \times \mathcal{A}^{(d)} \ni (f,A) \mapsto f\left[A \cap [0;1]^d\right] \in \mathcal{A}^{(k)}$ is $([\rho^d \to \rho^k] \times \psi_>^d, \psi_>^k)$–computable.
   ii) How about $(\psi_>, \psi_>)$–computability of the mapping $A \mapsto \sin[A]$ for arbitrary compact $A \subseteq \mathbb{R}$?

c) The lecture constructed a $\psi_>$–computable closed non-empty $A \subseteq [0;1]$ containing no computable point.

   i) Prove that every $\psi_>$–computable closed non-empty *convex* $C \subseteq [0;1]$ contains a computable point.
   ii) Generalize i) to $\psi_>^d$–computable closed non-empty convex $C \subseteq [0;1]^d$.
   Hint: induction on d using b i) and Exercise 13c).