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Computable Analysis

SS 2013, Exercise Sheet #3

EXERCISE 8:
Recall that the Halting Problem is a subset of integers since every Turing machine $\mathcal{M}$ can be assigned some code $\langle\mathcal{M}\rangle$, called its Gödel Index, such that the 'simulation' mapping $\mathbb{N}\times\mathbb{N}\ni(\langle\mathcal{M}\rangle,x)\mapsto\mathcal{M}(x)$ is computable. Now call $f:[0;1]\to\mathbb{R}$ Markov computable if there exists a Turing machine *converting* every Gödel index of some $\mathcal{M}$ computing a real number $x\in[0;1]$ into the Gödel index of some $\mathcal{N}$ computing $f(x)$.

a) Show that every computable $f:[0;1]\to\mathbb{R}$ is Markov computable.

b) Prove that Markov computability implies sequential computability (Exercise 6).

EXERCISE 9:

a) Formalize real number by approximation up to prescribable *relative* error. Make sure that 0 and 1 are relatively computable numbers. Compare the set of relatively computable numbers with $\mathbb{R}_c$. Is addition a relatively computable operation?

b) Prove that the open interval $(a,b)\subseteq\mathbb{R}$ is ρ–r.e.
iff a is $\rho_>$–computable and b is $\rho_<$–computable.

EXERCISE 10:
Call $\vec{u}\in\{0,1\}^*$ an initial segment of $\vec{v}\in\{0,1\}^*$ (and write "$\vec{u}\sqsubseteq\vec{v}$") if there exists $\vec{w}\in\{0,1\}^*$ with $\vec{v}=\vec{u}\circ\vec{w}$. Similarly write "$\vec{u}\sqsubseteq\bar{v}$" if $\bar{v}=\vec{u}\circ\bar{w}$ for $\bar{v}\in\{0,1\}^\omega$ and some $\bar{w}\in\{0,1\}^\omega$. For $\bar{\sigma}\in\{0,1\}^\omega$ abbreviate $\bar{\sigma}|_{\leq n}:=(\sigma_1,\ldots,\sigma_n)\in\{0,1\}^n$. We say that $f:\{0,1\}^*\to\{0,1\}^*$ is monotone if it holds

$$\forall\vec{u},\vec{v}:\quad \vec{u}\sqsubseteq\vec{v}\ \Rightarrow\ f(\vec{u})\sqsubseteq f(\vec{v})\ .$$

a) Suppose f is also unbounded on $\bar{\sigma}\in\{0,1\}^\omega$ in that the length $|f(\bar{\sigma}|_{\leq n})|$ is unbounded in n. Show that there exists precisely one $\bar{\tau}\in\{0,1\}^\omega$ (denoted by $\bar{\tau}=\sup_n f(\bar{\sigma}|_{\leq n})$) with $\forall n: f(\bar{\sigma}_{\leq n})\sqsubseteq\bar{\tau}$.

b) Suppose f is monotone. Prove that the following function $f_\omega:\subseteq\{0,1\}^\omega\to\{0,1\}^\omega$ is continuous:

$$\mathrm{dom}(f_\omega)=\{\bar{\sigma}\mid f\text{ unbounded on }\bar{\sigma}\},\qquad f_\omega:\bar{\sigma}\mapsto\sup_n f(\bar{\sigma}|_{\leq n})$$

c) Suppose $F:\subseteq\{0,1\}^\omega\to\{0,1\}^\omega$ is continuous.
Construct some monotone f with $f_\omega\big|_{\mathrm{dom}(F)}=F$.