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Computable Analysis

SS 2013, Exercise Sheet #2

EXERCISE 5:
The lecture constructed a computable function $f \in C^1[0;1]$ with uncomputable derivative.
Prove that every computable $f \in C^2[0;1]$ has a computable derivative!

EXERCISE 6:
Call $f : [0;1] \to \mathbb{R}$ sequentially computable if, for every computable sequence (x_m) in $[0;1]$, $f(x_m)$ is a computable sequence.

a) Show that every computable $f : [0;1] \to \mathbb{R}$ is sequentially computable.

b) Let (x_m) be a computable sequence such that $\{m : x_m < 1\}$ is infinite. Prove that there exists a computable, strictly increasing integer sequence $(m_k)_k$ containing all m with $x_m < 1$ and satisfying $x_{m_k} < 2$ for all k.

c) Recall from the lecture the computable sequence $(x_m) \in [0;1]$ such that $\{m : x_m \neq 0\} = H$, the Halting problem.
Now let $f : [0;1] \to \mathbb{R}$ be sequentially computable and (x_m) a computable sequence in $[0;1]$ with computable limit x_0. Conclude that $f(x_m)$ converges to $f(x_0) \in \mathbb{R}_c$.

d) Let $f : [0;1] \to \mathbb{R}$ be sequentially computable. Prove that, to every computable $x \in [0;1]$ and $\varepsilon, \delta > 0$, there exists some $q \in \mathbb{Q} \subseteq \mathbb{R}_c$ with $|x - q| < \delta$ and $|f(x) - f(q)| < \varepsilon$. Hint: Consider the sequence $f(q)_{q \in \mathbb{Q} \cap [0;1]}$ and recall that *in*equality is semi-decidable.

e) Suppose $x_0 \in [0;1]$ is computable and let (x_m) denote a (not necessarily computable) sequence in $\mathbb{R}_c$ with $|x_m - x_0| \leq 2^{-m-1}$ and $|f(x_m) - y| > 2\varepsilon$ for all m. Prove that there exists a computable rational sequence (q_n) with $|x_0 - q_n| \leq 2^{-n}$ and $|f(q_n) - y| > \varepsilon$.

f) Let $f : [0;1] \to \mathbb{R}$ be sequentially computable. Conclude that the restriction $f|_{\mathbb{R}_c}$ is continuous.

EXERCISE 7:
Call $f : [0;1] \to \mathbb{R}$ naively computable if there exists an algorithm converting any convergent rational sequence $q_n \to x \in [0;1]$ into a convergent rational sequence $p_m \to f(x)$.

a) Show that every computable $f : [0;1] \to \mathbb{R}$ is naively computable.

b) Prove that the Heaviside function is not naively computable.

c) More generally every naively computable $f : [0;1] \to \mathbb{R}$ is continuous.