Gaifman's theorem

$\varphi(\mathbf{x}) \in \mathrm{FO}(\sigma)$ is an *ℓ-local formula* if
$\varphi(\mathbf{x}) \equiv \varphi^\ell(\mathbf{x}) := [\varphi(\mathbf{x})]^{N^\ell(\mathbf{x})}$ (relativisation to $N^\ell(\mathbf{x})$),
i.e., for all $\mathfrak{A}, \mathbf{a}$: $\mathfrak{A}, \mathbf{a} \models \varphi$ iff $\mathfrak{A} \restriction N^\ell(\mathbf{a}), \mathbf{a} \models \varphi$

a *basic ℓ-local sentence* is an FO-sentence of the form

$$\varphi = \exists x_1 \ldots \exists x_m \big(\bigwedge_{i<j} d(x_i, x_j) > 2\ell \ \wedge \ \bigwedge_i \psi^\ell(x_i)\big)$$

for some ℓ-local formula $\psi^\ell(x) \in \mathrm{FO}_1(\sigma)$

NB: the following is a theorem of classical model theory

Gaifman's theorem

for any relational signature σ, every $\varphi(\mathbf{x}) \in \mathrm{FO}(\sigma)$
is logically equivalent to a boolean combination of
local formulae and basic local sentences

for a b&f proof of Gaifman's theorem:

the rank of the basic local sentence
$\varphi = \exists x_1 \ldots \exists x_m \big(\bigwedge_{i<j} d(x_i, x_j) > 2\ell \ \wedge \ \bigwedge_i \psi^\ell(x_i)\big)$ is $(\ell, \mathrm{qr}(\psi), m)$

definition

$\mathfrak{A}, \mathbf{a}$ and $\mathfrak{B}, \mathbf{b}$ are (ℓ, q, m)-*Gaifman-equivalent*, $\mathfrak{A}, \mathbf{a} \equiv^\ell_{q,m} \mathfrak{B}, \mathbf{b}$, if

- $\mathfrak{A}, \mathbf{a}$ and $\mathfrak{B}, \mathbf{b}$ satisfy the same ℓ-local formulae of quantifier rank up to q;
- $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences in ranks (ℓ', q', m') for $\ell \leqslant \ell$, $q' \leqslant q$, $m' \leqslant m$

lemma

if $\mathfrak{A}$ and $\mathfrak{B}$ are $(L, Q, m+n)$-Gaifman-equivalent for sufficiently large L, Q, $\mathbf{a}_0 \in A^n, \mathbf{b}_0 \in B^n$, then $(I_k)_{k \leqslant m} \colon \mathfrak{A}, \mathbf{a}_0 \simeq_m \mathfrak{B}, \mathbf{b}_0$

where, for suitable (ℓ_k, q_k), I_k consists of all partial isomorphisms $p = \mathbf{a} \mapsto \mathbf{b}$, $|p| \leqslant m + n - k$, s.t. $\mathfrak{A} \restriction N^{\ell_k}(\mathbf{a}), \mathbf{a} \equiv_{q_k} \mathfrak{B} \restriction N^{\ell_k}(\mathbf{b}), \mathbf{b}$

FMT expressive completeness results: examples

- modal logic $\mathrm{ML}(\subseteq \mathrm{FO})$ is expressively complete for first-order properties (of elts) invariant under bisimulation equivalence: $\mathrm{FO}/{\sim} \equiv \mathrm{ML}$ classically & FMT & a new proof

- $\exists$-$\mathrm{FO} \subseteq \mathrm{FO}$ is expressively complete for first-order properties of finite unions of finite successor chains that are preserved under extensions: a restricted FMT version of Łos–Tarski

- $\exists$-$\mathrm{FO}_{\mathrm{pos}} \subseteq \mathrm{FO}$ is expressively complete for first-order properties that are preserved under homomorphisms within wide classes of finite structures closed under disjoint union & substructures: a restricted FMT version of Lyndon–Tarski–Rossman

FO expressive completeness: classical vs. FMT

(i) $\varphi \in \mathrm{FO}$ preserved under $\mathfrak{A} \rightsquigarrow \mathfrak{B}$:
for all relevant $\mathfrak{A} \rightsquigarrow \mathfrak{B}$, $\mathfrak{A} \models \varphi \;\Rightarrow\; \mathfrak{B} \models \varphi$

(ii) $\varphi \equiv \varphi' \in \mathcal{L} \subseteq \mathrm{FO}$ over all relevant structures

(ii)' $\varphi \in \mathrm{FO}$ preserved under $\mathcal{L}$-transfer $\Rightarrow_{\mathcal{L}}$
over all relevant structures

(ii)" $\varphi \in \mathrm{FO}$ preserved under some approximation $\Rightarrow^{\ell}_{\mathcal{L}}$
based on finite index equivalence $\equiv^{\ell}_{\mathcal{L}}$ for $\ell = \ell(\varphi)$
over all relevant structures

preservation: (ii) $\Rightarrow$ (i) / **expressive completeness:** (i) $\Rightarrow$ (ii)
only for preservation, the classical version implies FMT version

classically can use (ii)' $\Rightarrow$ (ii) by compactness (!)

non-classical arguments rather rely on (ii)" $\Rightarrow$ (ii)