Fraïssé limits

theorem (Fraïssé)

if K is an $\simeq$-closed class of finite σ-structures in finite relational σ with (HP), (JEP), (AP), and with arbitrarily large finite members, then there is a countable σ-structure $\mathfrak{A}$ s.t.

- K is the class of finite substructures of $\mathfrak{A}$ $\quad (K = \mathrm{age}(\mathfrak{A}))$
- K is strongly ω-homogeneous:
  every finite partial isomorphism extends to an automorphism

this *Fraïssé limit of K* is unique up to $\simeq$; its FO-theory $\mathrm{Th}(\mathfrak{A})$

- is ω-categorical
- and has quantifier-elimination

so that $\mathfrak{A}$ is also the unique countable atomic as well as the unique countable ω-saturated model of $\mathrm{Th}(\mathfrak{A})$

random structures, Fagin's theorem, 0–1 laws

idea: consider the asymptotic behaviour for $n \to \infty$

of probabilities $\quad \mu_n(\varphi) := \dfrac{|\mathrm{Mod}_n(\varphi)|}{|\mathrm{Mod}_n(\top)|}$

that a uniformly random σ-structure of size n on $\{1, \dots, n\}$ satisfies φ, e.g. for $\varphi \in \mathrm{FO}_0(\sigma)$

- L has a limit law if $\mu(\varphi) := \lim_{n\to\infty}\mu_n(\varphi)$ exists for all $\varphi \in L$
- L has a 0–1 law if moreover $\mu(\varphi) \in \{0, 1\}$

theorem (Fagin)

for all finite relational signatures σ, $\mathrm{FO}_0(\sigma)$ has the 0–1 law; and the *almost sure theory* $\{\varphi \in \mathrm{FO}_0(\sigma) \colon \mu(\varphi) = 1\}$ is satisfiable and ω-categorical (hence complete), and coincides with the theory of the Fraïssé limit of the class of all finite σ-structures

example: the random graph/Rado graph

for $\sigma = \{E\}$, consider not the class of all (finite) σ-structures, but the class of all (finite) simple graphs, i.e., models of

$\varphi_0 := \forall x \forall y (\neg Exx \wedge Exy \leftrightarrow Eyx)$

- $\mathrm{FMod}_n(\varphi_0)$ has as its Fraïssé limit the *random graph* or *Rado graph* $\mathfrak{R}$
- its theory is axiomatised by the set of the φ_0-compatible extension axioms $\mathrm{Ext}[\varphi_0]$
- $\varphi \in \mathrm{Ext}[\varphi_0]$ is almost surely true in finite φ_0-models:

$$\mu(\varphi) := \frac{|\mathrm{Mod}_n(\varphi_0 \wedge \varphi)|}{|\mathrm{Mod}_n(\varphi_0)|} = 1$$

$\Rightarrow$ $\mathrm{Th}(\mathfrak{R}) = \mathrm{Ext}[\varphi_0]^{\models}$ is the almost sure theory of graphs

some further results on asymptotic probabilities

- full analogues of Fagin's result for graphs can be obtained for any *parametric* class of finite σ-structures, e.g., the class of finite tournaments ($\rightarrow$ exercises)
- FO has a 0–1 law over every class K of arbitrarily large finite graphs with (HP), (JEP) and (AP), but the almost sure theory need not coincide with the theory of the Fraïssé limit, nor need it be ω-categorical
  $\rightarrow$ results and survey in [Kolaitis–Prömel–Rothschild 87]
- for a signature σ with at least one non-unary relation symbol, finite σ-structures are almost surely rigid; the same holds true, e.g., of finite graphs, and even w.r.t. the asymptotic measure based on no. of $\simeq$-types rather than no. of realisations on $[n]$
  $\rightarrow$ e.g. [Ebbinghaus–Flum: Finite Model Theory]