ω-homogeneous structures

$\mathfrak{A}$ ω-homogeneous:

for $\mathfrak{A}, \mathbf{a} \equiv \mathfrak{A}, \mathbf{a}'$ and $a \in A$ there exists $a' \in A$ s.t. $\mathfrak{A}, \mathbf{a}a \equiv \mathfrak{A}, \mathbf{a}'a'$;
equivalently: finite partial elementary self-maps form b&f system

- atomic structures are ω-homogeneous,
  ω-saturated structures are ω-homogeneous
- for countable ω-homogeneous $\mathfrak{A}, \mathfrak{B}$: $\mathfrak{A} \equiv \mathfrak{B} \not\Rightarrow \mathfrak{A} \simeq \mathfrak{B}$
  unless $\mathfrak{A}$ and $\mathfrak{B}$ even realise the same n-types, f.a. $n \in \mathbb{N}$

thm

every countable theory T has countable ω-homogeneous models

thm

for countable structure $\mathfrak{A}$ in countable signature: $\mathfrak{A}$ is ω-saturated if, and only if, $\mathfrak{A}$ is both universal and ω-homogeneous

Fraïssé limits

from specification of suitable class K of finite (sub-)structures
$\rightsquigarrow$ generic countable limit structure & ω-categorical FO-theory

intuition: saturation/homogeneity w.r.t. isomorphic embeddings of members of K as substructures

examples: $(\mathbb{Q}, <)$ and the random graph (Rado graph)

some relevant *universal algebraic properties*
of $\simeq$-closed class K of (finite) σ-structures:

- hereditary property **(HP)**: closure under substructures
- joint embedding property **(JEP)**: any two members of K embed isomorphically into a common third member of K
- amalgamation property **(AP)**: find diamond completions for isomorphic embeddings within K

Fraïssé limits

theorem (Fraïssé)

if K is an $\simeq$-closed class of finite σ-structures in finite relational σ with (HP), (JEP), (AP), and with arbitrarily large finite members, then there is a countable σ-structure $\mathfrak{A}$ s.t.

- K is the class of finite substructures of $\mathfrak{A}$ $\quad (K = \mathrm{age}(\mathfrak{A}))$
- K is strongly ω-homogeneous:
  every finite partial isomorphism extends to an automorphism

this *Fraïssé limit of K* is unique up to $\simeq$; $\quad$ its FO-theory $\mathrm{Th}(\mathfrak{A})$

- is ω-categorical
- and has quantifier-elimination

so that $\mathfrak{A}$ is also the unique countable atomic as well as the unique countable ω-saturated model of $\mathrm{Th}(\mathfrak{A})$

random structures, Fagin's theorem, 0–1 laws

idea: consider the asymptotic behaviour for $n \to \infty$

of probabilities $\quad \mu_n(\varphi) := \dfrac{|\mathrm{Mod}_n(\varphi)|}{|\mathrm{Mod}_n|}$

that a uniformly random σ-structure of size n on $\{1, \ldots, n\}$ satisfies φ, e.g. for $\varphi \in \mathrm{FO}_0(\sigma)$

- L has a limit law, if $\mu(\varphi) := \lim_{n\to\infty}\mu_n(\varphi)$ exists for all $\varphi \in L$
- L has a 0–1 law, if moreover $\mu(\varphi) \in \{0, 1\}$

theorem (Fagin)

for all finite relational signatures σ, $\mathrm{FO}_0(\sigma)$ has the 0–1 law; and the *almost sure theory* $\{\varphi \in \mathrm{FO}_0(\sigma) \colon \mu(\varphi) = 1\}$ is satisfiable and ω-categorical (hence complete), and coincides with the theory of the Fraïssé limit of the class of all finite σ-structures