realising and omitting types

question: which types must be realised?

a model of T *omits* $p \in S_n(T)$ if p is not realised in $\mathfrak{A}$
it omits a partial type Φ if it omits every p in Φ

omitting types theorem

for satisfiable theory $T \subseteq \mathrm{FO}_0(\sigma)$ in a countable signature σ,
and a partial type $\Phi(\mathbf{x}) \subseteq \mathrm{FO}_n(\sigma)$ of T:

if Φ is not isolated in $S_n(T)$ by any formula $\varphi(\mathbf{x}) \in \mathrm{FO}_n(\sigma)$,
then T has a countable model that omits Φ

NB: a complete theory T must realise every isolated (partial) type

non-isolation of Φ means that every basic open
set in $S_n(T)$ is compatible with some violation of Φ

saturation properties

$\mathfrak{A}$ κ-saturated: every type with fewer than κ many
parameters from A is realised in $\mathfrak{A}$

$\mathfrak{A}$ saturated: every type with fewer than $|A|$ many
parameters from A is realised in $\mathfrak{A}$

- every structure possesses κ-saturated elementary extensions
- for saturated structures in cardinality κ:
  elementary equivalence implies isomorphism
- FO-theories may not have any saturated models
  (there may be too many types with parameters)

theorem (countable ω-saturated models)

a complete countable theory T has a countable saturated model if,
and only if, all $S_n(T)$ are countable

I.5 Countable models

investigate countable FO-theories T w.r.t. the richness of the class of their countable models:
$\longrightarrow$ **criteria w.r.t. to the type spaces $\mathbf{S_n(T)}$**

- many or (very) few countable models: ω-categoricity
- countable models with special properties:
  - atomic & prime models (just isolated types)
  - ω-saturated & universal (all types)
  - ω-homogeneous models (highly symmetric)

investigate a universal-algebraic approach to generic countable structures $\longrightarrow$ **Fraïssé limits**

ω-categoricity

T is ω-categorical if it has precisely one countably infinite model, up to $\simeq$

theorem (Engeler, Ryll-Nardzewski, Svenonius)

t.f.a.e. for any satisfiable theory $T \subseteq \mathrm{FO}_0(\sigma)$ in a countable signature σ:

(i) T is ω-categorical

(ii) $S_n(T)$ finite for all n

NB: $S_n(T)$ finite if, and only if, $\mathrm{FO}_n/\!\equiv_T$ finite, if, and only if, all n-types of T are isolated

$\rightarrow$ totally disconnected & compact spaces

the one extreme: countable atomic models

$\mathfrak{A}$ atomic: only isolated types of $\mathrm{Th}(\mathfrak{A})$ are realised

- countable atomic structures are minimal in the sense of being prime models of their theories, admitting elementary embeddings into every other model: this property characterises atomic models
- countable atomic models are unique up to $\simeq$ within their elementary equivalence class

thm

a complete countable theory T has countable atomic models if, and only if, the isolated types are dense in all $S_n(T)$

the other extreme: countable saturated models

- countable saturated structures are maximal in the sense of being universal among the countable models of their theory w.r.t. elementary embeddings
- moreover, countable saturated structures are rich in the sense of homogeneity: every finite elementary partial self-map extends to an automorphism

together these properties provide a characterisation

- countable saturated structures are unique up to $\simeq$ within their elementary equivalence class

thm

a complete countable theory T has a countable saturated model if, and only if, all $S_n(T)$ are countable