I.3: Back&Forth (aside/review)

proviso: treat relational signatures only!

$\mathrm{Part}(\mathfrak{A},\mathfrak{B})$: the set of all partial isomorphisms
$p \colon \mathfrak{A}{\restriction}\mathrm{dom}(p) \simeq \mathfrak{B}{\restriction}\mathrm{image}(p)$ (allowing $p = \emptyset$)

notation: $p \colon \mathbf{a} \mapsto \mathbf{b}$ for finite p and tuple $\mathbf{a}$ enumerating $\mathrm{dom}(p)$

NB: $(p \colon \mathbf{a} \mapsto \mathbf{b}) \in \mathrm{Part}(\mathfrak{A},\mathfrak{B})$ iff $\mathfrak{A},\mathbf{a} \equiv_0 \mathfrak{B},\mathbf{b}$
qfr-free FO-equivalence

back&forth extensions:

$p \in \mathrm{Part}(\mathfrak{A},\mathfrak{B})$ has b&f extensions in $I \subseteq \mathrm{Part}(\mathfrak{A},\mathfrak{B})$ if

forth: for all $a \in A$ there is some $p' \in I$,
such that $p \subseteq p'$ and $a \in \mathrm{dom}(p')$;

back: for all $b \in B$ there is some $p' \in I$,
such that $p \subseteq p'$ and $b \in \mathrm{image}(p')$;

back&forth systems

- $I \subseteq \mathrm{Part}(\mathfrak{A},\mathfrak{B})$ is a b&f system if
  every $p \in I$ has b&f extensions in I
- $(I_k)_{k \leqslant m}$ or $(I_k)_{k \in \mathbb{N}}$, where $I_k \subseteq \mathrm{Part}(\mathfrak{A},\mathfrak{B})$,
  are b&f systems if every $p \in I_{k+1}$ has b&f extensions in I_k

game intuition: b&f systems encode (non-deterministic)
winning strategies for second player in pebble games

- a flat b&f system I provides responses
  in infinite game in every position $p \in I$
- a stratified b&f system (I_k) provides responses that
  are safe for k further rounds from positions in $p \in I_k$

$\longrightarrow$ Ehrenfeucht–Fraïssé

levels of back&forth equivalence

m-isomorphy, $\mathfrak{A}, \mathbf{a} \simeq_m \mathfrak{B}, \mathbf{b}$:

$(p\colon \mathbf{a} \mapsto \mathbf{b}) \in I_m$ for some b&f system $(I_k)_{k \leqslant m}$
second player has winning strategy in
m-round game from $\mathfrak{A}, \mathbf{a}; \mathfrak{B}, \mathbf{b}$

finite isomorphy, $\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{fin}} \mathfrak{B}, \mathbf{b}$:

$(p\colon \mathbf{a} \mapsto \mathbf{b}) \in I_k$ for all k in some b&f system $(I_k)_{k \in \mathbb{N}}$
winning strategy for second player in every finite game
$\simeq_{\mathrm{fin}}$ is the common refinement of all the $\simeq_m$

partial isomorphy, $\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{part}} \mathfrak{B}, \mathbf{b}$:

$(p\colon \mathbf{a} \mapsto \mathbf{b}) \in I$ for some b&f system I
winning strategy for second player in infinite game

natural levels of equivalence supporting intuitive game arguments;
model-theoretic interest: relationships with levels of logical equivalence

review: Ehrenfeucht–Fraïssé theorem

recall levels of FO-equivalence: $\mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$
equivalence w.r.t. FO-formulae up to qfr-rk m

$\mathfrak{A}, \mathbf{a} \equiv \mathfrak{B}, \mathbf{b}$ iff $\mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$ for all $m \in \mathbb{N}$

Ehrenfeucht–Fraïssé Theorem

for *finite* relational σ and
σ-structures $\mathfrak{A}, \mathfrak{B}$ with parameters $\mathbf{a} \in A^n, \mathbf{b} \in B^n$:

- $\mathfrak{A}, \mathbf{a} \simeq_m \mathfrak{B}, \mathbf{b}$ iff $\mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$
- $\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{fin}} \mathfrak{B}, \mathbf{b}$ iff $\mathfrak{A}, \mathbf{a} \equiv \mathfrak{B}, \mathbf{b}$

NB: finiteness and relational nature of σ matters; it implies that $\simeq_m$ and $\equiv_m$ have finite index over the class of all σ-structures with n parameters

I.4 Types: logic and topology

n-type: (complete) specification of properties of an *n*-tuple by a set of formulae in free variables $\mathbf{x} = (x_1, \ldots, x_n)$,

or: a maximally consistent subset $p = p(\mathbf{x}) \subseteq \mathrm{FO}_n(\sigma)$

$$\mathrm{tp}^{\mathfrak{A}}(\mathbf{a}) := \{\varphi(\mathbf{x}) \in \mathrm{FO}_n(\sigma) \colon \mathfrak{A}, \mathbf{a} \models \varphi\},$$

the (complete) type of $\mathbf{a} \in A^n$ in σ-structure $\mathfrak{A}$, is the unique type p *realised* by the tuple $\mathbf{a}$ in $\mathfrak{A}$

variations:

- type $\mathrm{tp}^{\mathfrak{A}}_C(\mathbf{a})$ with parameters $C \subseteq A$ in $\mathfrak{A}_C$
- type of a theory T: realised/realisable in models $\mathfrak{A} \models T$
- partial *n*-type: any satisfiable $\Phi(\mathbf{x}) \subseteq \mathrm{FO}_n(\sigma)$; can be identified with the set of (complete) types $p \supseteq \Phi$

type spaces of a theory

for a satisfiable theory $T \subseteq \mathrm{FO}_0(\sigma)$,

$S_n(T)$ is the set of all *n*-types of *n*-tuples in models $\mathfrak{A} \models T$,

or: the set of all maximally consistent sets $T \subseteq \Phi(\mathbf{x}) \subseteq \mathrm{FO}_n(\sigma)$

NB: partial types are treated as special subsets of $S_n(T)$

$\mathbf{S_n(T)}$ as a topological space

with topology induced by the basis of open sets
$O_\varphi := \{p \in S_n(T) \colon \varphi \in p\}$

the Stone space of the theory T or of the boolean algebra of $\mathrm{FO}_n(\sigma)/{\equiv_T}$ (logical equivalence in T)

- Hausdorff and totally disconnected (the O_φ are clopen)
- **compact!**