application: Löwenheim–Skolem–Tarski

lemma ($\preccurlyeq$-criterion)

$A \subseteq B$ in σ-structure $\mathfrak{B}$ is the universe of an elementary substructure $\mathfrak{A} \preccurlyeq \mathfrak{B}$ (i.e., $\mathfrak{B}{\restriction}A \preccurlyeq \mathfrak{B}$) if, and only if

for every $\varphi(\mathbf{x}, x) \in \mathrm{FO}(\sigma)$ and $\mathbf{a}$ over A:
ex. $b \in B$ s.t. $\mathfrak{B}, \mathbf{a}, b \models \varphi \quad \Rightarrow \quad$ ex. $a \in A$ s.t. $\mathfrak{B}, \mathbf{a}, a \models \varphi$

theorem (Löwenheim–Skolem–Tarski)

for any σ-structure $\mathfrak{B}$ and $A_0 \subseteq B$,
there is some A, $A_0 \subseteq A \subseteq B$, such that:

- $\mathfrak{A} := \mathfrak{B}{\restriction}A \preccurlyeq \mathfrak{B}$
- $|A| \leqslant \max(\omega, |A_0|, |\mathrm{FO}(\sigma)|)$

NB: obtain new proof of Löwenheim–Skolem $\downarrow$ as a corollary

elementary chains: Tarski union property

limits of chains

a family of σ-structures $(\mathfrak{A}_\alpha)_{\alpha\in\lambda}$ indexed by an ordinal λ is

- a chain if $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ for all $\alpha \in \beta \in \lambda$
- an elementary chain if $\mathfrak{A}_\alpha \preccurlyeq \mathfrak{A}_\beta$ for all $\alpha \in \beta \in \lambda$

the limit of the chain $(\mathfrak{A}_\alpha)_{\alpha\in\lambda}$ is the unique σ-structure $\mathfrak{A} := \bigcup_\alpha \mathfrak{A}_\alpha$ on $\bigcup_\alpha A_\alpha$ for which $\mathfrak{A}_\alpha \subseteq \mathfrak{A}$ for all $\alpha \in \lambda$

elementary chain lemma (Tarski union property, TUP)

for any elementary chain $(\mathfrak{A}_\alpha)_{\alpha\in\lambda}$ with limit $\mathfrak{A} := \bigcup_\alpha \mathfrak{A}_\alpha$:

$\mathfrak{A}_\alpha \preccurlyeq \mathfrak{A}$ for all $\alpha \in \lambda$;

hence, any elementary class is closed under limits of elementary chains

applications of elementary chain constructions

preservation under chain limits:

while every $\varphi \in \mathrm{FO}$ is preserved under limits of elementary chains,
$\varphi \in \forall^*\exists^*\text{-}\mathrm{FO}$ (the prenex $\forall^*\exists^*$ fragment of FO)
is preserved under arbitrary unions of chains, and in fact,
t.f.a.e. for $\varphi \in \mathrm{FO}(\sigma)$:

(i) φ is preserved under limits of chains

(ii) $\varphi \equiv \varphi' \in \forall^*\exists^*\text{-}\mathrm{FO}(\sigma)$

φ preserved under limits (unions) of chains:

for any chain of σ-structures $(\mathfrak{A}_\alpha)_{\alpha<\lambda}$,
if $\mathfrak{A}_\alpha \models \varphi$ for all $\alpha < \lambda$, then $\mathfrak{A} \models \varphi$ for the limit $\mathfrak{A} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$

further examples of expressive completeness results

positive FO is preserved in homomorphic images,
positive existential FO under arbitrary homomorphisms ...

Łos–Lyndon–Tarski Theorems

(A) t.f.a.e. for $\varphi \in \mathrm{FO}(\sigma)$:

(i) φ is preserved under surjective homomorphisms

(ii) $\varphi \equiv \varphi' \in \mathrm{FO}_{\mathrm{pos}}(\sigma)$ (the positive fragment of FO)

(B) t.f.a.e. for $\varphi \in \mathrm{FO}(\sigma)$:

(i) φ is preserved under homomorphisms

(ii) $\varphi \equiv \varphi' \in \exists\text{-}\mathrm{FO}_{\mathrm{pos}}(\sigma)$ (existential positive fragment of FO)

Robinson consistency

yet another powerful application of elementary chains proves the

Robinson consistency theorem

in signatures τ_1, τ_2 and $\tau_0 := \tau_1 \cap \tau_2$:

if $\Phi_i \subseteq \mathrm{FO}_0(\tau_i)$ are such that

- Φ_0 is a complete theory (in $\mathrm{FO}_0(\tau_0)$), and
- $\Phi_1 \supseteq \Phi_0$ and $\Phi_2 \supseteq \Phi_0$ are both satisfiable,

then also $\Phi_1 \cup \Phi_2$ is satisfiable.

corollaries: Craig interpolation and Beth

corollary: Craig interpolation

if $\varphi_1 \models \varphi_2$ for $\varphi_i \in \mathrm{FO}(\tau_i)$, then there is some $\chi \in \mathrm{FO}(\tau_1 \cap \tau_2)$ such that $\varphi_1 \models \chi$ and $\chi \models \varphi_2$

corollary: Beth definability

implicit FO-definability implies explicit FO-definability

terminology: a relation $R \notin \sigma$ is *implicitly defined* by $\Sigma(R) \subseteq \mathrm{FO}_0(\sigma \cup \{R\})$ if, for renaming $R \rightsquigarrow R'$ by fresh R',

$$\Sigma(R) \cup \Sigma(R') \models \forall \mathbf{x}(R\mathbf{x} \leftrightarrow R'\mathbf{x});$$

an *explicit definition* (relative to Σ) then has the form

$$\Sigma \models \forall \mathbf{x}(R\mathbf{x} \leftrightarrow \xi(\mathbf{x})) \text{ for some } \xi(\mathbf{x}) \in \mathrm{FO}(\sigma)$$