Łos Theorem

Let $\mathfrak{A} := \prod_i \mathfrak{A}_i \,/\, \mathcal{U}$ be an ultraproduct of a family $(\mathfrak{A}_i)_{i \in I}$ of σ-structures $\mathfrak{A}_i$ w.r.t. an ultrafilter $\mathcal{U}$ on I.

Then, for any $\varphi(\mathbf{x}) = \varphi(x_1, \ldots, x_n) \in \mathrm{FO}_n(\sigma)$, and for any $\mathbf{a} = (a_1, \ldots, a_n) \in (\prod_i A_i)^n$:

$$\boxed{\mathfrak{A} \models \varphi\big[([a_1], \ldots, [a_n])\big] \quad \text{iff} \quad [\![\varphi(\mathbf{a})]\!] \in \mathcal{U}}$$

NB: $[\![\varphi(\mathbf{a})]\!] = \big\{ i \in I \colon \mathfrak{A}_i \models \varphi[\mathbf{a}(i)] \big\}$ serves as a set-valued semantic valuation over $\prod_i \mathfrak{A}_i$ and

"truth in $\prod_i \mathfrak{A}_i \,/\, \mathcal{U}$ is truth in $\mathcal{U}$-many components"

compactness via ultra-products

idea: for given $\Phi \subseteq \mathrm{FO}_0(\sigma)$, find I and ultrafilter $\mathcal{U}$ on I

together with map $s \colon \Phi \longrightarrow \mathcal{U}$, $\varphi \longmapsto s_\varphi$

such that, f.a. $i \in I$, the subset $\Phi_i := \{\varphi \in \Phi \colon i \in s_\varphi\} \subseteq \Phi$ is finite

then, for a family of models $\mathfrak{A}_i \models \Phi_i$, for $i \in I$:

(Łos) $\quad \prod_i \mathfrak{A}_i \,/\, \mathcal{U} \models \varphi \quad \text{iff} \quad [\![\varphi]\!] \in \mathcal{U}$,

and $\prod_i \mathfrak{A}_i \,/\, \mathcal{U} \models \varphi$ for every $\varphi \in \Phi$, since $[\![\varphi]\!] \supseteq s_\varphi \in \mathcal{U}$

…and suitable I and $\mathcal{U}$ can be found (NB: multiple uses of AC)

compactness via ultra-products: the countable case

for countable $\Phi = \{\varphi_n : n \in \mathbb{N}\} \subseteq \mathrm{FO}_0(\sigma)$,
can use any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$

as $\mathcal{U}$ extends the Frechet filter (!), can use $\Phi_i := \{\varphi_n : n \leqslant i\}$
and $\mathfrak{A}_i \models \Phi_i$ for all $i \in \mathbb{N}$ guarantees that $\prod \mathfrak{A}_i \,/\, \mathcal{U} \models \Phi$:

$[\![\varphi_n]\!] \supseteq \{i \in \mathbb{N} : i \geqslant n\} \in \mathcal{U}$

corollary: a saturation property

for σ-structures $\mathfrak{A}$ in at most countable σ
and $\Phi(x) = \{\varphi_n(x) : n \in \mathbb{N}\} \subseteq \mathrm{FO}_1(\sigma)$
such that $\mathfrak{A} \models \exists x \bigwedge_{n \leqslant i} \varphi$ for all i:

there is some $a \in A^{\mathbb{N}}$ such that $\mathfrak{A}^{\mathbb{N}}/\mathcal{U}, [a] \models \Phi$

compactness via ultraproducts: the general case

for $\Phi \subseteq \mathrm{FO}_0(\sigma)$ of cardinality κ, $\Phi = \{\varphi_\alpha : \alpha \in \kappa\}$,

use $I := \mathcal{P}_{\mathrm{fin}}(\kappa) = \{i \in \mathcal{P}(\kappa) : \ i \text{ finite } \}$
together with $s \colon \alpha \longmapsto s_\alpha := \{i \in I : \alpha \in i\}$,

and ultrafilter $\mathcal{U}$ on I with $\mathcal{U} \supseteq \mathcal{B}$
where $\mathcal{B} := \{s_\alpha : \alpha \in \kappa\}$ has f.i.p.

- if every finite $\Phi_0 \subseteq \Phi$ is satisfiable, then
  $\Phi_i := \{\varphi_\alpha : \alpha \in i\}$ is satisfiable for all $i \in I$
- if $\mathfrak{A}_i \models \Phi_i$ for all $i \in I$, then $\prod \mathfrak{A}_i \,/\, \mathcal{U} \models \Phi$

I.2 Elementary maps and chains

review: **relationships between σ-structures**

isomorphy & isomorphisms: $\pi\colon \mathfrak{A} \simeq \mathfrak{B}$ (or $\pi\colon \mathfrak{A}, \mathbf{a} \simeq \mathfrak{B}, \mathbf{b}$)

substructure/extension relationship: $\mathfrak{A} \subseteq \mathfrak{B}$

isomorphic embeddings: $\pi\colon \mathfrak{A} \simeq \mathfrak{A}' \subseteq \mathfrak{B}$

elementary embeddings $\pi\colon \mathfrak{A} \longrightarrow_{\mathrm{el}} \mathfrak{B}$:

$\pi\colon A \to B$ such that f.a. $\mathbf{a} \in A^n$: $\mathfrak{A}, \mathbf{a} \equiv \mathfrak{B}, \pi(\mathbf{a})$

cf. weaker notion of isomorphic embeddings

elementary substructure/extension relationship $\mathfrak{A} \preccurlyeq \mathfrak{B}$:

$A \subseteq B$ (and $\mathfrak{A} \subseteq \mathfrak{B}$) with elementary inclusion map $\iota\colon \mathfrak{A} \to \mathfrak{B}$

related notions for partial maps

partial isomorphisms $\pi \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$:

$\pi\colon \mathrm{dom}(\pi) \subseteq A \longrightarrow \mathrm{image}(\pi) \subseteq B$ such that f.a. $\mathbf{a} \in \mathrm{dom}(\pi)^n$
and all atomic $\alpha \in \mathrm{FO}_n(\sigma)$: $\mathfrak{A}, \mathbf{a} \models \varphi \;\Leftrightarrow\; \mathfrak{B}, \pi(\mathbf{a}) \models \varphi$

partial elementary maps:

$\pi \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ such that f.a. $\mathbf{a} \in \mathrm{dom}(\pi)^n$: $\mathfrak{A}, \mathbf{a} \equiv \mathfrak{B}, \pi(\mathbf{a})$

partial isomorphisms and partial elementary maps arise
as approximations, and in back&forth arguments
(cf. Ehrenfeucht–Fraïssé techniques)

extensions & elementary extensions

for σ-structure $\mathfrak{A}$ put $\sigma_A := \sigma \dot\cup \{c_a : a \in A\}$,
and let $\mathfrak{A}_A$ be the natural σ_A expansion of $\mathfrak{A}$

algebraic (or quantifier-free) diagram:

$\mathrm{D_{alg}}(\mathfrak{A}) := \{\varphi(\mathbf{c_a}) : \mathfrak{A} \models \varphi[\mathbf{a}],\ \varphi \in \mathrm{FO}(\sigma) \text{ qfr-free } \}$

elementary diagram:

$\mathrm{D}(\mathfrak{A}) = \mathrm{D_{el}}(\mathfrak{A}) := \{\varphi(\mathbf{c_a}) : \mathfrak{A} \models \varphi[\mathbf{a}],\ \varphi \in \mathrm{FO}(\sigma) \ \} \ \big[\equiv \mathrm{Th}(\mathfrak{A}_A)\ \big]$

up to isomorphism,

- the extensions of $\mathfrak{A}$ are the σ-reducts of models of $\mathrm{D_{alg}}(\mathfrak{A})$
- the elementary extensions of $\mathfrak{A}$ are
  the σ-reducts of models of $\mathrm{D_{el}}(\mathfrak{A})$

application: a first 'preservation theorem'

theorem (Tarski)

t.f.a.e. for $\varphi(\mathbf{x}) \in \mathrm{FO}(\sigma)$:

(i) φ is preserved under substructures

(ii) $\varphi \equiv \varphi' \in \forall\text{-}\mathrm{FO}(\sigma)$ (the universal fragment of FO)

NB: (ii) $\Rightarrow$ (i) is easy by syntactic induction on $\varphi \in \forall\text{-}\mathrm{FO}(\sigma)$
(i) $\Rightarrow$ (ii) is a non-trivial *expressive completeness* result