relational recursion: fixpoint logics

$\varphi(X, \mathbf{x}) \in \mathrm{FO}_k(\sigma \cup \{X\})$ with k-ary X and matching $\mathbf{x}$ induces operation on $\mathcal{P}(A^k)$, uniformly across all $\mathfrak{A} \in \mathrm{Fin}(\sigma)$:

$$\begin{array}{rcl} \mathcal{F}^{\mathfrak{A}}_{\varphi} \colon \mathcal{P}(A^k) & \longrightarrow & \mathcal{P}(A^k) \\ P & \longmapsto & \{\mathbf{a} \in A^k : \mathfrak{A}, P, \mathbf{a} \models \varphi\} \end{array}$$

easy to see: if $\varphi(X, \mathbf{x})$ is X-positive, this operation is monotone (preservation result/classically only: matching expressive completeness)

natural extensions of FO, esp. for FMT, provide recursion mechanisms based on such definable operations

- **least fixpoint logic LFP** has least and greatest fixpoints for positive/monotone operations
- **partial fixpoint logic PFP** has fixpoints for arbitrary operations (with default $\emptyset$)

capturing results with order

thm (Immerman–Vardi)

Ptime $\equiv$ LFP over linearly ordered structures

i.e., t.f.a.e. for every class $\mathcal{C} \subseteq \mathrm{Fin}(\sigma)$ of linearly ordered σ-structures:

(i) $\mathcal{C} \subseteq \mathrm{Fin}(\sigma)$ is decidable in NP

(ii) $\mathcal{C}$ is definable within $\mathrm{Fin}(\sigma)$ by a sentence of $\mathrm{LFP}(\sigma)$

thm (Abiteboul–Vianu)

Pspace $\equiv$ PFP over linearly ordered structures

remarks: order is crucial, simple fixpoints over FO suffice
model-checking in Ptime/Pspace is obvious
expressive completeness: coding & fixpoint recursion

proof ideas: coding over linearly ordered structures

Ptime/LFP:

encode run $(C_t)_{t<n^k}$ of DTM on input $\mathfrak{A} = (n, <, \dots)$ as a relation $R \subseteq A^{3k+2}$, which is definable as the least fixpoint of monotone/X-positive inductive process that allows to add new X-entries (for C_{t+1}) based on existing X-entries (for C_t)

Pspace/PFP:

produce sequence of n^k-bounded configurations (C_t) of DTM on input $\mathfrak{A} = (n, <, \dots)$ as stages of FO-definable operation mapping X (for C_t) to $\mathcal{F}_\varphi^{\mathfrak{A}}$ (for C_{t+1})

so that termination within space bound n^k yields encoding of final configuration as PFP fixpoint

in both cases, get 'normal form' with single (unnested) application of fixpoint operators to first-order formulae

fixpoint operations and k-variable logic

lemma

for $\varphi(X, \mathbf{x}) \in \mathrm{FO}^k_k(\sigma)$ with X and $\mathbf{x}$ of arity k:
$\mathcal{F}_\varphi^{\mathfrak{A}}$ is compatible with $\simeq^k_\infty$ and preserves closure under $\simeq^k_\infty$

$\rightsquigarrow$ resulting least or partial fixpoints are closed under $\simeq^k_\infty$

and fixpoint iteration over $\mathfrak{A}$ is faithfully represented over the linearly ordered k-pebble invariant $\mathfrak{I}^k(\mathfrak{A}, \mathbf{a})$, where LFP captures Ptime and PFP captures Pspace

lemma

the linearly ordered k-pebble invariant $\mathfrak{I}^k(\mathfrak{A}, \mathbf{a})$ itself is LFP-definable (interpretable) over $\mathfrak{A}$ (without ordering)

easy: complement of $\simeq^k_\infty$ is a least fixed point of stages $\not\simeq_i$ based on induction step $\mathbf{x} \not\simeq_{i+1} \mathbf{x}'$ if $\mathbf{x} \not\simeq_i \mathbf{x}' \vee \bigvee_{j \in [k]} \exists y \forall y' \big(\mathbf{x}\frac{y}{j} \not\simeq_i \mathbf{x}'\frac{y'}{j}\big) \vee \dots$

Abiteboul–Vianu theorem

question:
what does the relationship between LFP and PFP over not necessarily ordered finite structures tell us about Ptime vs. Pspace?

clearly Pspace collapses to Ptime if, and only if, $\mathrm{LFP} \equiv \mathrm{PFP}$ over the class of all linearly ordered finite σ-structures, even just for $\sigma = \{<, P\}$ with one unary predicate P

surprisingly strong link:
the collapse of Pspace to Ptime implies that $\mathrm{LFP} \equiv \mathrm{PFP}$ over the class of all finite σ-structures (for any σ)

thm (Abiteboul–Vianu)

Pspace = Ptime if, and only if, $\mathrm{LFP} \equiv \mathrm{PFP}$ (in FMT throughout)

Abiteboul–Vianu thm: proof idea

suppose $(*)$ Pspace = Ptime, and let $\mathcal{C} = \mathsf{FMod}(\varphi)$, $\varphi \in \mathrm{PFP}(\sigma)$

choose k such that all subformulae of φ are preserved under $\simeq^k_\infty$

then, uniformly across all $\mathfrak{A} \in \mathsf{Fin}(\sigma)$:

- $\varphi^{\mathfrak{A}}$ can be evaluated on $\mathfrak{I}^k(\mathfrak{A})$;
- this evaluation on $\mathfrak{I}^k(\mathfrak{A})$ is in Pspace, hence in Ptime by $(*)$;
- as $\mathfrak{I}^k(\mathfrak{A})$ is linearly ordered, the outcome is LFP-definable over $\mathfrak{I}^k(\mathfrak{A})$ by the Immerman–Vardi theorem;
- as $\mathfrak{I}^k(\mathfrak{A})$ is LFP-interpretable over $\mathfrak{A}$, $\mathcal{C}$ is LFP-definable

in fact:

$$\mathrm{LFP} \equiv \bigcup_k \mathrm{Ptime}(\mathfrak{I}^k)$$

$$\mathrm{PFP} \equiv \bigcup_k \mathrm{Pspace}(\mathfrak{I}^k)$$

and the collapse in size from $\mathfrak{A}$ to $\mathfrak{I}^k(\mathfrak{A})$ for unordered $\mathfrak{A}$ accounts for the possible deviation from the ordered case