monadic second-order logic MSO

MSO extends FO by quantification over subsets of the universe correspondingly allowing set variables X along with element variables x

$\mathfrak{A}, \mathbf{P}, \mathbf{a} \models \exists X \varphi(\mathbf{X}, X, \mathbf{x})$ if $\mathfrak{A}, \mathbf{P}, P, \mathbf{a} \models \varphi$ for some $P \subseteq A$

MSO is a natural level expressiveness for many purposes, e.g., the following are MSO-definable (but notFO-definable):

- connectedness of (finite) graphs
- even length of finite linear orderings
- k-colourability of (finite) graphs
- transitive closures of binary relations (how?)

the natural extension of the first-order Ehrenfeucht–Fraïssé game to MSO allows the first player to choose, in each round, between (first-order) element moves and (second-order) set moves

Büchi's theorem

associate (non-empty) words over finite alphabet Σ with linearly ordered word models in signature $\sigma_\Sigma = \{<\} \cup \{P_a : a \in \Sigma\}$

a Σ-language $L \subseteq \sigma^*$ is *regular* if it is recognised by a finite automaton iff (Myhill–Nerode) it is a union of equivalence classes of some right-invariant equivalence relation of finite index on Σ^*

thm (Büchi–Elgot–Trakhtenbrot)

a class of Σ-word models is $\mathrm{MSO}(\sigma_\Sigma)$-definable if, and only if, the corresponding Σ-language is regular

a *capturing result* in the spirit of *descriptive complexity theory*, giving a machine-independent, logical characterisation of a complexity class of algorithmic problems in terms of definability in a logic