Model Theory

guiding question: **what can/cannot be expressed in specified logical formalisms?**

- analysis of expressive power and semantics
- construction, analysis and classification of models
- methods from logic, universal algebra, combinatorics, ...

classical model theory: study expressiveness of FO and its fragments over the class of all (finite and infinite) structures

non-classical model theory: study expressiveness of specific logics over specific classes of structures

> **e.g., finite model theory: only finite models count, lose FO compactness/proof calculi, but ...**

Model Theory

has connections with diverse areas of mathematics and computer science

in mathematics: algebra, universal algebra, set-theoretic constructions, combinatorics, discrete mathematics, logic and topology, decidability issues and algorithms, ...

in theoretical computer science: decidability and complexity, descriptive complexity, specification&verification, model checking, modelling and reasoning about finite or infinite systems, database theory, constraint satisfaction, ...

examples

groups are structures of the form $\mathfrak{G} = (G, \circ^{\mathfrak{G}}, e^{\mathfrak{G}})$ satisfying

$$\left.\begin{array}{l}\text{(G1) associativity of } \circ \\ \text{(G2) } e \text{ (right) neutral for } \circ \\ \text{(G3) existence of (right) inverses for } \circ\end{array}\right\} \mathrm{FO}(\{\circ, e\})$$

the class of all **groups**, $\mathrm{Mod}(\{(\mathrm{G1}), (\mathrm{G2}), (\mathrm{G3})\})$, is closed under

- homomorphic images
- direct products
- chain limits

but not under passage to substructures, unless . . .

. . . and how can we tell from the axioms?

. . . conversely, what axioms befit which classes of structures?

preservation and expressive completeness theorems

examples

of algorithmic issues

Which logics $\mathcal{L}$ (e.g., fragments $\mathcal{L} \subseteq \mathrm{FO}$) are decidable for SAT (over certain classes $\mathcal{C}$ of structures; or, e.g., through fmp)?

- **decidability and complexity of $\mathbf{SAT}(\mathcal{L}, \mathcal{C})$, $\mathbf{FINSAT}(\mathcal{L}, \mathcal{C})$**

What is the relationship between complexity and logical definability over certain classes $\mathcal{C}$ of structures?

- **descriptive complexity theory**

Do certain (undecidable) classes of problems admit syntactic representations in terms of tailor-made logics?

- **expressive completeness of logics $\mathcal{L}$ for specific purposes**

examples

compactness for first-order logic FO

$\Phi \subseteq \mathrm{FO}(\sigma)$ satisfiable if (and only if)
every *finite* subset $\Phi_0 \subseteq \Phi$ is satisfiable

first proof (Introduction to Mathematical Logic):
via completeness, i.e., via detour through syntax
(finiteness property obvious for consistency)

alternative proof (Model Theory, universal algebra):
can construct model of Φ from models of all finite $\Phi_0 \subseteq \Phi$
using ultra-products for model construction

model construction techniques in relation to
logical definability, **expressiveness**, **FO theories**

terminology and basic notions:

the classes of all **σ-structures**, for signatures σ,

$$\mathfrak{A} = \big(A, (f^{\mathfrak{A}})_{f \in \mathrm{fctn}(\sigma)}, (R^{\mathfrak{A}})_{R \in \mathrm{rel}(\sigma)}, (c^{\mathfrak{A}})_{c \in \mathrm{const}(\sigma)}\big)$$

with universe/domain $A \neq \emptyset$ and interpretations

$\mathfrak{I}^{\mathfrak{A}}(f) = f^{\mathfrak{A}} \colon A^n \to A$ for n-ary function symbol f

$\mathfrak{I}^{\mathfrak{A}}(R) = R^{\mathfrak{A}} \subseteq A^n$ for n-ary relation symbol R

$\mathfrak{I}^{\mathfrak{A}}(c) = c^{\mathfrak{A}} \in A$ for constant symbol c

support **natural notions from universal algebra:**

homomorphisms, isomorphisms, automorphisms, ...

substructures/extensions, products, quotients, chain limits, ...

reducts/expansions, ...

terminology and basic notions:

syntax (for first-order logic)**:**

σ-terms (T_σ) and σ-formulae ($\mathrm{FO}(\sigma)$), free variables,

$\mathrm{FO}_n(\sigma) = \{\varphi \in \mathrm{FO}(\sigma) \colon \mathrm{free}(\varphi) \subseteq \{x_1, \ldots, x_n\}\}$,
shorthand $\varphi = \varphi(x_1, \ldots, x_n)$ for $\varphi \in \mathrm{FO}_n(\sigma)$

$\mathrm{FO}_0(\sigma) = \{\varphi \in \mathrm{FO}(\sigma) \colon \mathrm{free}(\varphi) = \emptyset\}$, σ-sentences,

theories $T \subseteq \mathrm{FO}_0(\sigma)$ $^{(*)}$

semantics (w.r.t. σ-structures and assignments $\mathfrak{A}, \beta$ or $\mathfrak{A}, \mathbf{a}$)

satisfaction relation: $\mathfrak{A}, \beta \models \Phi$, $\mathfrak{A}, \mathbf{a} \models \varphi(\mathbf{x})$, $\mathfrak{A} \models \varphi[\mathbf{a}]$

semantic relation of consequence, $\psi \models \varphi$, $\Phi \models \varphi$

semantic notions of satisfiability, validity, ...

$^{(*)}$ satisfiable theories $T \subseteq \mathrm{FO}_0(\sigma)$ often assumed closed under $\models$

I: Elements of Classical Model Theory

- compactness via ultra-products, Łos Theorem
- elementary substructures & extensions, elementary chains, examples of classical 'preservation theorems', Robinson consistency, Craig interpolation, Beth's theorem
- topology of types, compactness & saturation properties, countable models, realising & omitting types, ω-categoricity, Fraïssé limits

I.1 Compactness via ultra-products

direct product of family of σ-structures $(\mathfrak{A}_i)_{i\in I}$

$\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i = \big(\prod_i A_i, (R^{\mathfrak{A}}), (f^{\mathfrak{A}}), (c^{\mathfrak{A}})\big)$

with 'component-wise' interpretations of $R, f, c \in \sigma$
over $A := \prod_i A_i = \big\{(a(i))_{i\in I} \colon a(i) \in A_i \text{ f.a. } i \in I\big\}$

reduced product of $(\mathfrak{A}_i)_{i\in I}$ w.r.t. filter $\mathcal{F}$ on I: $\prod_i \mathfrak{A}_i \,/\, \mathcal{F}$

obtained as natural quotient of direct product $\prod_i \mathfrak{A}_i$
w.r.t. filter-equivalence $\sim_{\mathcal{F}}$ on $\prod_i A_i$:

for $a = (a(i))_{i\in I},\ a' = (a'(i))_{i\in I} \in \prod_i A_i$:

$a \sim_{\mathcal{F}} a'$ if $\{i \in I \colon a(i) = a'(i)\} \in \mathcal{F}$
agreement in $\mathcal{F}$-many components

filters and ultrafilters

filter $\mathcal{F}$ (on infinite set I):

- $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(I) \setminus \{\emptyset\}$
- $\mathcal{F}$ closed under supersets: $s \in \mathcal{F}, s \subseteq s' \subseteq I \Rightarrow s' \in \mathcal{F}$
- $\mathcal{F}$ closed under (finite) intersections: $s, s' \in \mathcal{F}, \Rightarrow s \cap s' \in \mathcal{F}$

ultrafilters are maximal filters:

characterised by the condition that for every $s \in \mathcal{P}(I)$,
precisely one of s or $\bar{s} = I \setminus s$ is a member of $\mathcal{F}$

existence: AC implies that every collection of subsets
of I with the *finite intersection property* (fip)
can be extended to an ultrafilter

examples: principal ultrafilters (boring): $\mathcal{F}_a := \big\{s \subseteq I \colon a \in s\big\}$;
in contrast, the Frechet-filter $\mathcal{F}$ of co-finite subsets of I, esp. of $\mathbb{N}$,
has $\bigcap \mathcal{F} = \emptyset$, and every ultrafilter extension of $\mathcal{F}$ is non-principal

Łos Theorem

Let $\mathfrak{A} := \prod_i \mathfrak{A}_i \,/\, \mathcal{U}$ be an ultraproduct of a family $(\mathfrak{A}_i)_{i \in I}$ of σ-structures $\mathfrak{A}_i$ w.r.t. an ultrafilter $\mathcal{U}$ on I.

Then, for any $\varphi(\mathbf{x}) = \varphi(x_1, \ldots, x_n) \in \mathrm{FO}_n(\sigma)$, and for any $\mathbf{a} = (a_1, \ldots, a_n) \in (\prod_i A_i)^n$:

$$\mathfrak{A} \models \varphi\big[([a_1], \ldots, [a_n])\big] \quad \text{iff} \quad [\![\varphi(\mathbf{a})]\!] \in \mathcal{U}$$

NB: $[\![\varphi(\mathbf{a})]\!] = \big\{ i \in I : \mathfrak{A}_i \models \varphi[\mathbf{a}(i)] \big\}$ serves as a set-valued semantic valuation over $\prod_i \mathfrak{A}_i$ and

"truth in $\prod_i \mathfrak{A}_i \,/\, \mathcal{U}$ is truth in $\mathcal{U}$-many components"

compactness via ultra-products

idea: for given $\Phi \subseteq \mathrm{FO}_0(\sigma)$, find I and ultrafilter $\mathcal{U}$ on I together with map $s \colon \Phi \longrightarrow \mathcal{U}$, $\varphi \longmapsto s_\varphi$

such that, f.a. $i \in I$, the subset $\Phi_i := \{\varphi \in \Phi : i \in s_\varphi\} \subseteq \Phi$ is finite

then, for a family of models $\mathfrak{A}_i \models \Phi_i$, for $i \in I$:

(Łos) $\quad \prod_i \mathfrak{A}_i \,/\, \mathcal{U} \models \varphi \quad$ iff $\quad [\![\varphi]\!] \in \mathcal{U}$,

and $\prod_i \mathfrak{A}_i \,/\, \mathcal{U} \models \varphi$ for every $\varphi \in \Phi$, since $[\![\varphi]\!] \supseteq s_\varphi \in \mathcal{U}$

... and suitable I and $\mathcal{U}$ can be found (NB: multiple uses of AC)