Classical and Non-Classical Model Theory

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Exercises No.9

Exercise 1 [Exercise 4.3 recycled]
Try to think of counter-examples for some of the classical results seen so far if we read them in the sense of finite model theory (fmt), where logical equivalence, consequence, preservation and definability only refer to finite structures as potential models.

(a) Consider the universal $\mathrm{FO}(\{<, S, \min, \max\})$-sentence φ_0 that says that $<$ is a linear ordering of the universe with minimal and maximal elements min and max and that S is a subset of the successor relation of $<$. Let φ_1 be a sentence asserting that S is the full successor relation for $<$. Check that $\varphi := \varphi_0 \wedge \neg\varphi_1$ is preserved in any substructure of any of its finite models, but not generally preserved under substructures. Can a condition that is equivalent with φ over all finite structures be expressed in universal FO? NB: this would imply that over finite models of φ_0, φ_1 would have to be expressible in existential FO (why?).

(b) Show that $<$-invariant FO-definability in restriction to just finite models can make a real difference. Can you think of an example of a property of finite structures that is $<$-invariantly definable but not without an auxiliary linear order $<$?
Hint: Ehrenfeucht–Fraïssé arguments show that the property of having an even number of atoms is not FO-definable over the class of finite boolean algebras.

(c) Try to find a counter-example to interpolation in fmt, i.e., sentences $\varphi_i \in \mathrm{FO}_0(\tau_i)$ for $i = 1, 2$ such that $\varphi_1 \rightarrow \varphi_2$ is valid over all finite structures, but such that there is no $\chi \in \mathrm{FO}_0(\tau_1 \cap \tau_2)$ for which $\varphi_1 \rightarrow \chi$ and $\chi \rightarrow \varphi_2$ would be valid over all finite structures.
Hint: set things up so that the desired interpolant would have to define evenness of finite $\emptyset$-structures (which is impossible by Ehrenfeucht–Fraïssé arguments).

(d) Similar to the previous example, find an example of a relation that is implicitly FO-definable but not explicitly FO-definable in restriction to all finite structures.

Exercise 2 [Gaifman distance]
For a finite relational signature σ, provide formulae that express the following:

(i) $d(x, y) \leqslant \ell$ for fixed $\ell \in \mathbb{N}$;

(ii) the relativisation of $\varphi(\mathbf{x}) \in \mathrm{FO}_n(\sigma)$ to the ℓ-neighbourhood of $\mathbf{x}$: $[\varphi(\mathbf{x})]^{N^\ell(\mathbf{x})}$.

For $\ell = 2^q - 1$, can you provide a formula $\varphi_q(x) \in \mathrm{FO}_1(\{E, P\})$ of quantifier rank q expressing, for undirected simple E-graphs with a unary P, that there is a P-node within distance ℓ of x?

Exercise 3 [locality and compactness (Exercise 5.5 recycled)]
Let $\varphi(x) \in \mathrm{FO}_1(\sigma)$ in relational signature σ be such that

$$\mathfrak{A} \models \varphi[a] \quad \text{iff} \quad \mathfrak{A}{\restriction}\mathrm{N}^\infty(a) \models \varphi[a]$$

where $\mathrm{N}^\infty(a)$ stands for the connected component of a in the Gaifman graph $G(\mathfrak{A})$. Show that φ must be ℓ-local for some $\ell \in \mathbb{N}$.

(a) give a compactness argument for this claim;

(b) give an Ehrenfeucht–Fraïssé argument to show the more specific claim for $\ell = 2^q - 1$ where q is the quantifier rank of φ (cf. the previous exercise for this bound).

Hint: for (a) one may play with conditions that imply the equivalence $\varphi(x) \leftrightarrow \varphi^U(x)$ between φ and its relativisation to the substructure whose universe is the interpretation of a new unary predicate $U \notin \sigma$; for (b), let $\mathfrak{B}, b := \mathfrak{A} \restriction \mathrm{N}^\ell(a), a$ where $\ell = 2^q - 1$: we want show that $\mathfrak{A}, a \models \varphi$ iff $\mathfrak{B}, b \models \varphi$; under the circumstances it suffices to show that the two structures obtained from $\mathfrak{A}, a$ and $\mathfrak{B}, b$ by adjoining the disjoint union of q copies each of $\mathfrak{A}$ and $\mathfrak{B}$, are q-isomorphic:

$$\underbrace{\mathfrak{A} \,\dot\cup\, \dots \,\dot\cup\, \mathfrak{A}}_{q} \;\dot\cup\; \mathfrak{A}, a \;\dot\cup\; \underbrace{\mathfrak{B} \,\dot\cup\, \dots \,\dot\cup\, \mathfrak{B}}_{q} \quad \simeq_q \quad \underbrace{\mathfrak{A} \,\dot\cup\, \dots \,\dot\cup\, \mathfrak{A}}_{q} \;\dot\cup\; \mathfrak{B}, b \;\dot\cup\; \underbrace{\mathfrak{B} \,\dot\cup\, \dots \,\dot\cup\, \mathfrak{B}}_{q};$$

for this, provide a suitable strategy for the second player.

Suggested Homework Exercises

Exercise 4 [non-expressibility over finite structures]
Show that the following properties are not first-order definable properties of finite structures (and contrast these proofs with classical non-definability results where appropriate).

(a) even cardinality of finite linear orderings or of finite successor chains.

(b) connectivity of finite graphs.

(c) planarity of finite graphs.

Exercise 5 [application of Hanf locality]
A graph is called k-connected, if it remains connected after removal of up to k vertices (ordinary connectivity is 0-connectivity). Show that, as a property of finite graphs and for fixed $k \in \mathbb{N}$, k-connectivity is not definable by any FO-sentence.

Exercise 6 [a Feferman–Vaught theorem]
Let σ be a (finite) relational signature and $\mathfrak{A}$ and $\mathfrak{B}$ be two (finite) σ-structures. The *disjoint union* $\mathfrak{A} \oplus \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ is defined to be the structure whose universe is the disjoint union $A \dot\cup B$ of the universes A and B (realised as $A \times \{0\} \cup B \times \{1\}$) and with $R^{\mathfrak{A} \oplus \mathfrak{B}} := R^{\mathfrak{A}} \dot\cup R^{\mathfrak{B}}$ for every $R \in \sigma$. Show that for $m \in \mathbb{N}$ and σ-structures $\mathfrak{A}_i, \mathfrak{B}_i$:

$$\mathfrak{A}_1 \equiv_m \mathfrak{A}_2, \mathfrak{B}_1 \equiv_m \mathfrak{B}_2 \;\Rightarrow\; \mathfrak{A}_1 \oplus \mathfrak{B}_1 \equiv_m \mathfrak{A}_2 \oplus \mathfrak{B}_2.$$

Which of the finiteness assertions matter? Does σ need to be relational?