Martin Otto
Kord Eickmeyer

Summer 2013

Exercises No.8

Exercise 1 [warm-up: Fraïssé limits]
Show that a class $\mathcal{C}$ of finite relational σ-structures with (HP) satisfies (AP) if, and only if, for all $n \geqslant 1$ and every quantifier-free $(n+1)$-type $\theta_{n+1}(x_1, \dots, x_n, x_{n+1})$ that can be realised in some $\mathfrak{C} \in \mathcal{C}$ and its restriction θ_n to its first n variables: if $\mathfrak{B}, \mathbf{b} \models \theta_n$ for $\mathfrak{B} \in \mathcal{C}$ then there is some $\mathfrak{B} \subseteq \mathfrak{B}' \in \mathcal{C}$ with $b' \in B'$ such that $\mathfrak{B}', \mathbf{b}b \models \theta_{n+1}$.
Show that the class of all finite partial orderings possesses a Fraïssé limit.
Can you think of a concrete representation of this limit structure?

Exercise 2 [asymptotic probabilities and the random graph]
Determine the asymptotic probability (or its non-existence, as the case may be) of the following properties of finite graphs.

(i) even cardinality
(ii) existence of at least one isolated node
(iii) connectedness
(iv) diameter 2
(v) planarity
(vi) k-colourability ($k > 1$ fixed)
(vii) triangle-freeness
(viii) existence of at least two nodes of the same degree
(ix) existence of a node of degree 17

Suggested Homework Exercises

Exercise 3 [explicit representations of random structures]
Show that the following structures on $\mathbb{N}$ are concrete representations of the Rado graph, i.e., $\mathfrak{R} \simeq (\mathbb{N}, E_i)$ for $i = 1, 2$:

(a) let $(p_n)_{n\in\mathbb{N}}$ be the enumeration of all primes in increasing order and let E_1 be the symmetric closure of the set of all pairs $(n, m) \in \mathbb{N}^2$ for which p_n divides m.
(b) let $[m]_2 = b_0(m) \ldots \in \{0, 1\}^*$ stand for the binary representation of the number $m \in \mathbb{N}$, starting with the least significant bit as bit $b_0(m)$, and let E_2 be the symmetric closure of the set of all pairs $(n, m) \in \mathbb{N}^2$ for which $b_n(m) = 1$.

Try to obtain a concrete representation of the Fraïssé limit for bipartite graphs (in signature $\{E, P\}$ with unary P for the colouring one of the partition sets).

Exercise 4 [Fagin's theorem for tournaments]
A tournament is a directed loop-free graph in which any two distinct vertices have precisely one of the two possible edge directions between them. Show that FO has a 0–1 law over the class of finite tournaments, and that the almost sure theory of tournaments is the ω-categorical theory of the Fraïssé limit of the class of finite tournaments.
Can you provide a concrete representation of the countable random tournament?