Classical and Non-Classical Model Theory

Martin Otto
Kord Eickmeyer
Summer 2013

Exercises No.7

Exercise 1 [countably many colours]
Let $\sigma = \{P_i : i \in \mathbb{N}\}$ be a signature consisting of a countable supply of unary predicates P_i, and let T be the FO(σ)-theory axiomatised by the following sentences, for all finite partial maps ρ from $\mathbb{N}$ to $\{0,1\}$:

$$\varphi_\rho := \exists x\big(\bigwedge\{P_i x \colon \rho(i) = 1\} \wedge \bigwedge\{\neg P_i x \colon \rho(i) = 0\}\big).$$

Show that T is a consistent and complete theory. Analyse $S_1(T)$ and discuss the class of countable models of T. Show that T cannot have any atomic models, but that all models of T are ω-homogeneous (and even strongly homogeneous). In which cardinalities does T have saturated models?

Suggested Homework Exercises

Exercise 2 [atomic and saturated countable models]
Show for all satisfiable, complete, countable theories T: if T has a countable saturated model, then it also has an atomic model.

Hint: any non-trivial open subset of $S_n(T)$ that does not contain any isolated type must have cardinality 2^ω (the cardinality of the complete binary tree); this (topo)logical argument uses an inductive binary splitting of an initial basis open set without O_φ isolated elements to embed the complete binary tree injectively.

Curious extra: if T has at most countably many countable models up to $\simeq$, then among them there must be an atomic and a saturated one; unless T is ω-categorical these two are distinct, but it is impossible that these are the only two isomorphism types (why?).

Exercise 3 [Fraïssé limits]
Which of the following classes K_i of finite structures possess a Fraïssé limit? Determine the limit structure and discuss its theory.

(i) K_1: finite graphs without n-cliques (for fixed $n \geqslant 3$)
(ii) K_2: finite planar graphs
(iii) K_3: finite bipartite graphs (in signature $\tau = \{E\}$)
(iv) K_4: finite bipartite coloured graphs (in signature $\tau = \{E, P\}$)
(v) K_5: finite equivalence relations
(vi) K_6: finite linear orderings
(vii) K_7: acyclic finite graphs