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Exercises No.6

Exercise 1 [countably many colours]
Let $\sigma = \{P_i : i \in \mathbb{N}\}$ be a signature consisting of a countable supply of unary predicates P_i, and let T be the FO(σ)-theory axiomatised by the following sentences, for all finite partial maps ρ from $\mathbb{N}$ to $\{0, 1\}$:

$$\varphi_\rho := \exists x \big(\bigwedge \{P_i x : \rho(i) = 1\} \wedge \bigwedge \{\neg P_i x : \rho(i) = 0\} \big).$$

Show that T is a consistent and complete theory; analyse $S_1(T)$; and discuss the class of countable models of T. In which cardinalities does T have saturated models?

Exercise 2 [type spaces]
Consider the FO-theories of $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$ and of its reduct $\mathfrak{N}{\restriction}\{<\}$. For each of these, discuss $S_1(T)$, the existence of countable saturated models, and of atomic models.

Exercise 3 [κ-saturated models: Prop. 4.12]
Let κ be an infinite regular cardinal (e.g., $\kappa = \omega$ or $\kappa = \omega_1$, the smallest uncountable ordinal). Show that any model $\mathfrak{A} \models T$ possesses an elementary extension $\hat{\mathfrak{A}} \succcurlyeq \mathfrak{A}$ that is κ-saturated. Hint: obtain $\hat{\mathfrak{A}}$ as a limit of an elementary chain $(\mathfrak{A}_\alpha)_{\alpha<\kappa}$ in which each $\mathfrak{A}_{\alpha+1}$ realises all types with fewer than κ parameters over $\mathfrak{A}_\alpha$; regularity of κ means that κ is not the union of fewer than κ smaller ordinals, whence every subset of $\bigcup_{\alpha<\kappa} A_\alpha$ of size $< \kappa$ must be contained in some A_α.

Exercise 4 [equivalence relations]

(a) Consider the FO-theory T of one equivalence relation $\sim$ and countably many constant symbols $(c_i)_{i\in\mathbb{N}}$ saying that the c_i are pairwise inequivalent and that every equivalence class is infinite. Determine the type space $S_1(T)$ and classify the countable models of T up to isomorphism.

(b) Investigate the completions and the type spaces of the (incomplete) theory of one equivalence relation $\sim$ (to be continued in Exercise 5 below).

Suggested Homework Exercises

Exercise 5 [equivalence relations]
Continue the analysis of the completions of the FO-theory of one equivalence relation $\sim$ from the previous exercise. A good starting point is the classification of all countable models up to isomorphism and up to elementary equivalence.
What are atomic and ω-saturated models?

Exercise 6 [back&forth in saturated models: Prop. 4.14]
Let κ be an infinite cardinal (e.g., $\kappa = \omega$ or $\kappa = \omega_1$, the smallest uncountable ordinal). Show that any two elementarily equivalent κ-saturated σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ are isomorphic.

Exercise 7 [extra for lovers of set theory: end extensions]
An elementary *end extension* of a model of set theory ZF, $\mathfrak{A} = (A, \in^{\mathfrak{A}})$, is a proper elementary extension $\mathfrak{B} \succneqq \mathfrak{A}$ in which no member of $A \subseteq B$ gets new elements in $\mathfrak{B}$, i.e., in which the extension of old sets remains unchanged in the sense that f.a. $a \in A$

$$\{b \in B \colon b \in^{\mathfrak{B}} a\} = \{b \in A \colon b \in^{\mathfrak{B}} a\} = \{b \in A \colon b \in^{\mathfrak{A}} a\} \subseteq A.$$

Show that every countable $\mathfrak{A} \models \mathrm{ZF}$ admits an elementary end extension.

Hint: The crucial step is this. For a new constant c put $T := \mathrm{D_{el}}(\mathfrak{A}) \cup \{\neg\, c \in c_a \colon a \in A\}$. We are looking for a model of T that omits all types of the form

$$p_a := \{x \in c_a\} \cup \{\neg\, x = c_d \colon d \in^{\mathfrak{A}} a\}.$$

It remains to show that none of these types is isolated in $S_1(T)$.
Show that $\varphi(x, c) \in \mathrm{FO}(\{\in, c\}_A)$ is consistent with T iff

$$\mathrm{D_{el}}(\mathfrak{A}) \models \forall z \exists y \exists x (\neg\, y \in z \wedge \varphi(x, y)).$$

From this, one can use standard set theoretic (ZF) reasoning in $\mathfrak{A}$ to show that, if $T \cup \{\varphi(x, c)\} \models x \in c_a$, then for some $d \in A$, $\varphi(c_d, c) \wedge c_d \in c_a$ is consistent with T, whence $\varphi(x, c)$ does not isolate p_a.