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Exercises No.5

Exercise 1 [review: Ehrenfeucht–Fraïssé]

(a) Review the argument that FO-formulae of quantifier rank m are preserved under m-isomorphism:

  (i) by syntactic induction, appealing to back&forth extensions in the quantification steps; and/or, alternatively,

  (ii) by arguing for a winning strategy for the first player in the m-round game from any position $\mathfrak{A}, \mathbf{a}; \mathfrak{B}, \mathbf{b}$ is which *not* $\mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$.

(b) Show that $\mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$ implies $\mathfrak{A}, \mathbf{a} \simeq_m \mathfrak{B}, \mathbf{b}$ for structures $\mathfrak{A}$ and $\mathfrak{B}$ in a finite relational signature σ: establish the back&forth conditions for

$$I_k := \big\{ (p \colon \mathbf{a} \mapsto \mathbf{b}) \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B}) \colon \mathfrak{A}, \mathbf{a} \equiv_k \mathfrak{B}, \mathbf{b} \big\}.$$

Provide counter-examples for critical claims in the case of infinite relational signatures, and discuss how to treat non-relational signatures.

Exercise 2 [back&forth arguments]

Show that any two countable dense linear orderings without endpoints are isomorphic. Conclude that the FO theory of dense linear orderings without endpoints is complete. Show that there are dense linear orderings of the same uncountable cardinality that are not isomorphic (though elementarily equivalent).
Extra: What is the situation for dense linear orderings without endpoints with a unary predicate for a dense subset?

Exercise 3 [Ehrenfeucht–Fraïssé: standard examples]

Determine (in typical cases that you can tackle) the degrees of equivalence between

(i) two naked sets ($\sigma = \emptyset$).

(ii) two finite linear orderings ($\sigma = \{<\}$).

(iii) two discrete linear orderings with first and last elements ($\sigma = \{<\}$).

(iv) two successor chains without first or last elements, possibly also allowing finite components that are directed cycles ($\sigma = \{E\}$).

Suggested Homework Exercises

Exercise 4 [partial isomorphy and saturation in ultrapowers]

Give examples of elementarily equivalent structures that are not partially isomorphic. Use insights analogous to those covered in Exercise 2.7 to show that for countable ultrapowers w.r.t. a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, elementary equivalence of the base structures implies partial isomorphy of their ultrapowers:

$$\text{for } \hat{\mathfrak{A}} := \mathfrak{A}^{\mathbb{N}}/\mathcal{U},\ \hat{\mathfrak{B}} := \mathfrak{B}^{\mathbb{N}}/\mathcal{U},\quad \mathfrak{A} \equiv \mathfrak{B} \ \Rightarrow\ \hat{\mathfrak{A}} \simeq_{\mathrm{part}} \hat{\mathfrak{B}}$$

Hint: in fact, the set of all finite (or even countable) elementary partial maps between $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{B}}$ is a back&forth system.

Exercise 5 [a locality property]
For this exercise consider graph-like structures with a finite relational vocabulary σ consisting of binary and unary relation symbols (think of edge- and vertex-coloured directed graphs). Let $\varphi(x) \in \mathrm{FO}(\sigma)$ be such that

$$\mathfrak{A} \models \varphi[a] \quad \text{iff} \quad \mathfrak{A}{\restriction}\mathrm{N}^{\infty}(a) \models \varphi[a]$$

where $\mathrm{N}^{\infty}(a)$ stands for the connected component of a in $\mathfrak{A}$.
Show that there must be some $\ell \in \mathbb{N}$ such that

$$\mathfrak{A} \models \varphi[a] \quad \text{iff} \quad \mathfrak{A}{\restriction}\mathrm{N}^{\ell}(a) \models \varphi[a]$$

where $\mathrm{N}^{\ell}(a)$ is the radius ℓ neighbourhood of a in $\mathfrak{A}$:

(a) give a compactness argument for this claim;

(b) give an Ehrenfeucht–Fraïssé argument to show the more specific claim for $\ell = 2^q - 1$ where q is the quantifier rank of φ.

Hint: for (a) one may play with conditions that imply the equivalence $\varphi(x) \leftrightarrow \varphi^U(x)$ between φ and its relativisation to the substructure whose universe is the interpretation of a new unary predicate $U \notin \sigma$;
for (b), let $\mathfrak{B}, b := \mathfrak{A}{\restriction}\mathrm{N}^{\ell}(a), a$, where $\ell = 2^q - 1$, and show that $\mathfrak{A}, a \models \varphi$ iff $\mathfrak{B}, b \models \varphi$; under the circumstances it suffices to show that the two structures obtained from $\mathfrak{A}, a$ and $\mathfrak{B}, b$ by adjoining the disjoint union of q copies each of $\mathfrak{A}$ and $\mathfrak{B}$, are q-isomorphic:

$$\underbrace{\mathfrak{A} \,\dot\cup \ldots \dot\cup\, \mathfrak{A}}_{q} \;\dot\cup\; \mathfrak{A}, a \;\dot\cup\; \underbrace{\mathfrak{B} \,\dot\cup \ldots \dot\cup\, \mathfrak{B}}_{q} \quad \simeq_q \quad \underbrace{\mathfrak{A} \,\dot\cup \ldots \dot\cup\, \mathfrak{A}}_{q} \;\dot\cup\; \mathfrak{B}, b \;\dot\cup\; \underbrace{\mathfrak{B} \,\dot\cup \ldots \dot\cup\, \mathfrak{B}}_{q};$$

for this, provide a strategy for the second player that allows her to maintain positions $p \colon \mathbf{a} \mapsto \mathbf{b}$, with k more rounds to play, in which the $N^{\ell(k)}$-neighbourhoods of $\mathbf{a}$ and $\mathbf{b}$, for $\ell(k) = 2^k - 1$, are linked by an isomorphism.