Classical and Non-Classical Model Theory

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Exercises No.4

Exercise 1 [interpolation from Robinson]
Fill in the details in the classical proof of the Craig Interpolation Theorem from the Robinson Consistency Theorem. Let $\varphi_1 \models \varphi_2$ for $\varphi_i \in \mathrm{FO}_0(\tau_i)$, let $\tau_0 := \tau_1 \cap \tau_2$ and assume there were no interpolant $\chi \in \mathrm{FO}_0(\tau_0)$ for this implication. In this situation, show that there would be some complete τ_0-theory Φ_0 such that both $\Phi_0 \cup \{\varphi_1\}$ and $\Phi_0 \cup \{\neg\varphi_2\}$ are satisfiable – contradicting Robinson consistency.

Hint: w.l.o.g. all signatures are finite and $\mathrm{FO}_0(\tau_0)$ can be enumerated as $(\psi_n)_{n\in\mathbb{N}}$; inductively select ψ_n or $\neg\psi_n$ into $\Phi_0^{(n)}$ so as to maintain the condition that there is no interpolant for $\bigwedge \Phi_0^{(n)} \wedge \varphi_1 \models \bigwedge \Phi_0^{(n)} \wedge \varphi_2$; then $\Phi_0 := \bigcup \Phi_0^{(n)}$ is as desired.

Suggested Homework Exercises

Exercise 2 [invariant definability]
Let $R \notin \sigma$, $\xi(R), \varphi(R) \in \mathrm{FO}_0(\sigma \cup \{R\})$. Then φ is R-invariant w.r.t. ξ if for any σ-structure $\mathfrak{A}$, any two expansions $(\mathfrak{A}, R_i) \models \xi(R)$ that interpret R in accordance with ξ satisfy $(\mathfrak{A}, R_1) \models \varphi$ iff $(\mathfrak{A}, R_2) \models \varphi$. (E.g., an *order-invariant* sentence is one that refers to an additional linear ordering but evaluates to the same truth value for any expansion by a total linear ordering.) Show that the class of those σ-structures $\mathfrak{A}$ that satisfy $\varphi(R)$ for some/any expansion $(\mathfrak{A}, R) \models \xi(R)$ is directly definable by a sentence $\varphi_0 \in \mathrm{FO}_0(\sigma)$.

Exercise 3 [very open preview: contrasting classical results with finite model theory]
Try to think of counter-examples for some of the classical results seen so far if we read them in the sense of finite model theory (fmt), where logical equivalence, consequence, preservation and definability only refer to finite structures as potential models.

(a) Consider the universal $\mathrm{FO}(\{<, S, \min, \max\})$-sentence φ_0 that says that $<$ is a linear ordering of the universe with minimal and maximal elements min and max and that S is a subset of the successor relation of $<$. Let φ_1 be a sentence asserting that S is the full successor relation for $<$. Check that $\varphi := \varphi_0 \wedge \neg\varphi_1$ is preserved in any substructure of any of its finite models, but not generally preserved under substructures. Can a condition that is equivalent with φ over all finite structures be expressed in universal FO? NB: this would imply that over finite models of φ_0, φ_1 would have to be expressible in existential FO (why?).

(b) Show that, in contrast with the previous exercise, R-invariant FO-definability in restriction to just finite models can make a real difference. Can you think of an example of a property of finite structures that is R-invariantly definable but not without the auxiliary R?
Hint: Ehrenfeucht–Fraïssé arguments show that the property of having an even number of atoms is not FO-definable over the class of finite boolean algebras.

(c) Try to find a counter-example to interpolation in fmt, i.e., sentences $\varphi_i \in \mathrm{FO}_0(\tau_i)$ for $i = 1, 2$ such that $\varphi_1 \rightarrow \varphi_2$ is valid over all finite structures, but such that there is no $\chi \in \mathrm{FO}_0(\tau_1 \cap \tau_2)$ for which $\varphi_1 \rightarrow \chi$ and $\chi \rightarrow \varphi_2$ would be valid over all finite structures.

Hint: set things up so that the desired interpolant would have to define evenness of finite $\emptyset$-structures (which is impossible by Ehrenfeucht–Fraïssé arguments).

(d) Similar to the previous example, find an example of a relation that is implicitly FO-definable but not explicitly FO-definable in restriction to all finite structures.