Classical and Non-Classical Model Theory

Martin Otto
Kord Eickmeyer
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Exercises No.3

Exercise 1 [warm-up: well-orderings]
Recall that the class of ordinals (in ZFC) is formed by those sets that are transitive (downward-closed) w.r.t. the $\in$-relation and well-ordered by the $\in$-relation; they form a complete system of representatives for all well-orderings up to $\simeq$. You may treat the following from a model-theoretic point of view (set-theoretically naive).

(i) The class of well-orderings is not first-order definable (not even Δ-elementary).

(ii) Any ordered sum (order-theoretic concatenation) of two well-orderings is a well-ordering. How about infinite ordered sums?

(iii) Is the class of well-orderings closed under limits of chains?

(iv) Is the class of ordinals closed under (set-)unions?

Exercise 2 [warm-up: variation on Tarski's thm, Thm 2.6/Lemma 2.12]
Show (the expressive completeness claim of) Tarski's theorem in the following variant, where $\Delta := \forall\text{-FO}_0[\sigma]$ with the associated notion of $\forall$-transfer $\mathfrak{A} \Rightarrow_\forall \mathfrak{B}$: any satisfiable $\varphi \in \text{FO}_0[\sigma]$ that is preserved under $\Rightarrow_\forall$ is equivalent to some $\varphi' \in \forall\text{-FO}[\sigma]$.

Exercise 3 [the $\forall^*\exists^*$ fragment of FO]
Consider the fragment $\Delta \subseteq \text{FO}_0[\sigma]$ consisting of all $\forall^*\exists^*$-sentences in $\text{FO}[\sigma]$, i.e., all prenex sentences of the form $\forall\mathbf{x}\exists\mathbf{y}\chi(\mathbf{x},\mathbf{y})$ with a quantifier-free kernel χ and define $\forall^*\exists^*$-transfer $\mathfrak{A} \Rightarrow_{\forall^*\exists^*} \mathfrak{B}$ accordingly.

(a) Show that $\mathfrak{A} \Rightarrow_{\forall^*\exists^*} \mathfrak{B}$ implies the existence of σ-structures $\mathfrak{A}' \equiv \mathfrak{A}$ and $\mathfrak{B}' \succcurlyeq \mathfrak{B}$ such that $\mathfrak{B} \subseteq \mathfrak{A}' \subseteq \mathfrak{B}'$.

(b) Show that preservation of $\varphi \in \text{FO}_0[\sigma]$ under limits of chains implies its preservation under $\Rightarrow_{\forall^*\exists^*}$.

Hints: for (a), work with FO-theories, algebraic and (the universal parts of) elementary diagrams as appropriate to specify suitable $\mathfrak{A}'_B$ and $\mathfrak{B}'_{A'}$;
towards (b), use the extension configuration from (a) repeatedly to construct an elementary ω-chain $\mathfrak{B} = \mathfrak{B}_0 \preccurlyeq \mathfrak{B}_1 \preccurlyeq \cdots$ with an interleaving ω-chain of structures $\mathfrak{A}_i \equiv \mathfrak{A}$,

$$\mathfrak{B}_0 \subseteq \mathfrak{A}_0 \subseteq \mathfrak{B}_1 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{B}_1 \subseteq \cdots$$

Then these chains have the same limits $\mathfrak{A}_\omega = \mathfrak{B}_\omega$. If $\mathfrak{A} \models \varphi$ and if φ is preserved under chains , then $\mathfrak{A}_\omega \models \varphi$ implies that also $\mathfrak{B} = \mathfrak{B}_0 \preccurlyeq \mathfrak{B}_\omega = \mathfrak{A}_\omega \models \varphi$.
NB: the topic of this exercise is pursued further in Exercise 4.

Suggested Homework Exercises

Exercise 4 [$\forall^*\exists^*$ preservation and limits of chains]
Continue from the basis of Exercise 3 above to show that the following are equivalent for any satisfiable $\varphi \in \mathrm{FO}_0[\sigma]$:

(i) φ is preserved under limits of chains;

(ii) $\varphi \equiv \varphi'$ where φ' is an $\forall^*\exists^*$-sentence.

Hint: for the expressive completeness argument adapt the proof idea from Lemma 2.12 and Exercise 2 above to the $\forall^*\exists^*$-fragment (which, although not closed under conjunction and disjunction literally, is closed in this sense up to logical equivalence).

Exercise 5 [Lyndon–Tarski Theorem]
Show that the following are equivalent for any satisfiable $\varphi \in \mathrm{FO}_0[\sigma]$:

(i) φ is preserved under (not neceassrily surjective) homomorphisms;

(ii) $\varphi \equiv \varphi'$ where φ' is positive existential ($\varphi' \in \exists_{\mathrm{pos}}\text{-FO}(\sigma)$).

Hint: for the expressive completeness argument adapt the proof idea from Thm 2.6 (Tarski's thm) to show that $\Phi \models \varphi$ where Φ is the set of all $\exists_{\mathrm{pos}}\text{-FO}(\sigma)$ consequences of φ. To this end, let $\mathfrak{A} \models \Phi$, and establish $\mathfrak{A} \models \varphi$ as follows.

(a) Let $\Psi := \mathrm{Th}(\mathfrak{A}) \cap \big\{\neg\exists\mathbf{x}\chi(\mathbf{x}) \colon \chi \text{ pos, qfr-free } \big\}$ and show that $\Psi \cup \{\varphi\}$ is satisfiable.

(b) For any $\mathfrak{B} \models \Psi \cup \{\varphi\}$, show that $\mathrm{Th}(\mathfrak{A}) \cup \mathrm{D}_{\mathrm{pos}}(\mathfrak{B})$ is statisfiable, where $\mathrm{D}_{\mathrm{pos}}(\mathfrak{B})$ is the positive (part of the) algebraic diagram of $\mathfrak{B}$. Conclude that there must be some $\mathfrak{A}' \equiv \mathfrak{A}$ to which $\mathfrak{B}$ can be mapped homomorphically. It follows that $\mathfrak{A} \models \varphi$.