Martin Otto
Kord Eickmeyer

Summer 2013

Exercises No.2

Exercise 1 [warm-up]
Let I be an index set and $(\mathfrak{A}_i)_{i \in I}$ a family of structures.

(i) Let $\mathcal{U}_j := \{s \subseteq I : j \in s\}$ be the principal ultrafilter generated by $\{j\}$. Then $\prod_i \mathfrak{A}_i \,/\, \mathcal{U}_j \cong \mathfrak{A}_j$.

(ii) For nonempty $s_0 \subseteq I$, let $\mathcal{F}_{s_0} := \{s \subseteq I : s_0 \subseteq s\}$ be the principal filter generated by s_0. Then $\prod_i \mathfrak{A}_i \,/\, \mathcal{F}_{s_0} \cong \prod_{i \in s_0} \mathfrak{A}_i$.

(iii) For the trivial filter $\mathcal{F} := \{I\}$ we have $\prod_i \mathfrak{A}_i \,/\, \mathcal{F} \cong \prod_i \mathfrak{A}_i$.

Exercise 2 [elementary embedding into ultrapowers]
For σ-structure $\mathfrak{A}$ and an ultrafilter $\mathcal{U}$ on I, let $\iota : A \to A^I \,/\, \mathcal{U}$ be the function

$$\begin{array}{rccl} \iota : & A & \longrightarrow & A^I \,/\, \mathcal{U} \\ & a & \longmapsto & [a^I] \end{array}$$

where $a^I := (a)_{i \in I}$ is the family with constant entry a. Show that ι is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{A}^I \,/\, \mathcal{U}$. In particular, $\mathfrak{A} \equiv \mathfrak{A}^I \,/\, \mathcal{U}$.

Exercise 3 [$\preccurlyeq$-criterion, Lemma 2.7]
Show that the following are equivalent for any subset $A \subseteq B$ of a σ-structure $\mathfrak{B}$:

(i) $\mathfrak{A} := \mathfrak{B} \restriction A$ is a σ-structure s.t. $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

(ii) for every $\varphi(\mathbf{x}, x) \in \mathrm{FO}(\sigma)$ and $\mathbf{a}$ over A: $\left\{ \begin{array}{ll} & \text{ex. } b \in B \text{ s.t. } \mathfrak{B} \models \varphi[\mathbf{a}, b] \\ \Rightarrow & \text{ex. } a \in A \text{ s.t. } \mathfrak{B} \models \varphi[\mathbf{a}, a]. \end{array} \right.$

Exercise 4 [countable algebraically closed field of characteristic zero]
Show the existence of countable, algebraically closed (respectively real closed) fields of characteristic zero. These cam be obtained as limits of suitable chains of substructures of suitable larger fields.

Suggested Homework Exercises

Exercise 5 [Lemma 2.5]
Prove the following assertions about the algebraic diagram $\mathrm{D}_{\mathrm{alg}}(\mathfrak{A})$ and the elementary diagram $\mathrm{D}_{\mathrm{el}}(\mathfrak{A})$ of a σ-structure $\mathfrak{A}$.

(a) T.f.a.e. for any σ-structure $\mathfrak{B}$:
  (i) there is an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{B}$
  (ii) $\mathfrak{B} = \mathfrak{B}_A \restriction \sigma$ for some $\mathfrak{B}_A \models \mathrm{D}_{\mathrm{alg}}(\mathfrak{A})$.

(b) T.f.a.e. for any σ-structure $\mathfrak{B}$:
  (i) there is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$
  (ii) $\mathfrak{B} = \mathfrak{B}_A \restriction \sigma$ for some $\mathfrak{B}_A \models \mathrm{D}_{\mathrm{el}}(\mathfrak{A})$.

Exercise 6 [preservation of ∀-FO under substructures]
Show by syntactic induction on $\varphi(\mathbf{x}) \in \forall\text{-FO}(\sigma)$ that universal FO-formulae are preserved under substructures, i.e., that for all $\mathfrak{A} \subseteq \mathfrak{B}$ and all $\mathbf{a}$ in $\mathfrak{A}$:

$$\mathfrak{B} \models \varphi[\mathbf{a}] \quad \Rightarrow \quad \mathfrak{A} \models \varphi[\mathbf{a}].$$

Give examples of common model classes that are closed under substructures and consider whether they admit an axiomatisation in $\forall\text{-FO}(\sigma)$ (note that this may depend on the choice of signatures; why?).

Exercise 7 [saturation property of countable ultrapowers]
Let σ be an at most countable signature, $\mathfrak{A}$ a σ-structure, and $\Phi(x) \subseteq \mathrm{FO}_1(\sigma)$ a set of formulae such that

$$\mathfrak{A} \models \exists x \bigwedge \Phi_0$$

for all finite $\Phi_0 \subseteq \Phi$. Then for all non-principal ultrafilters $\mathcal{U}$ on $\mathbb{N}$ there is some $(a_i)_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ such that

$$\mathfrak{A}^{\mathbb{N}} \,/\, \mathcal{U}, [(a_i)_{i\in\mathbb{N}}] \models \Phi.$$

What kind of density property does this imply for a countable ultrapower of $(\mathbb{Q}, <)$? Note how the construction of ultrapowers is compatible with expansions/reducts.