Classical and Non-Classical Model Theory

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Exercises No.13

Exercise 1 [warm-up: fixpoint formalisations]
Provide definitions of the following in LFP:

(a) the graph of addition as a 3-ary relation over the standard universes $(n, <)$, $n = \{0, \dots, n-1\}$ with the natural linear ordering, uniformly for $n \geqslant 1$.
(b) connectivity as a property of finite (or infinite) graphs.
(c) the set of winning positions in the k-pebble game over (the disjoint union of two) graphs, for fixed k.

Exercise 2 [on LFP and FO^k_∞]
Show that least fixed points over X-positive formulae in $\mathrm{FO}^k(\sigma \cup \{X\})$ may not be closed under $\simeq^k_\infty$ over finite structures if φ may have free first-order variables that serve as parameters in the fixpoint iteration.

Exercise 3 [Immerman–Vardi and Abiteboul–Vianu theorems]
Sketch the encoding of the runs of a Ptime or Pspace Turing machine that recognises some class $\mathcal{C}$ of linearly ordered finite graphs to obtain a recipe for defining that same class in LFP(E) or PFP(E), respectively.

Suggested Homework Exercise

Exercise 4 [fmp for FO^2]
Analyse the finite and exponential small model properties for $\mathrm{FO}^2(\sigma)$ for fixed finite relational σ along these lines:

(a) (relational skolemisation): every $\varphi \in \mathrm{FO}^2_0(\sigma)$ is satisfiability equivalent with some $\hat{\varphi} \in \mathrm{FO}^2_0(\hat{\sigma})$ of the following normal form

$$\hat{\varphi} = \forall x \forall y \psi_0(xy) \;\wedge\; \bigwedge_{i=1}^{m} \forall x \exists y \psi_i(xy)$$

for quantifier-free formulae ψ_i in an extended relational vocabulary $\hat{\sigma}$, where $\hat{\varphi}$ (and $\hat{\sigma}$) are linearly bounded in the size of φ (and polynomial time computable).

(b) (small models for $\hat{\varphi}$ in normal form): every satisfiable $\hat{\varphi} \in \mathrm{FO}^2_0$ as above possesses a model whose size is linearly bounded in the number of quantifier-free 2-types in signature $\hat{\sigma}$.

Hint for (b): starting from an arbitrary model $\mathfrak{A} \models \hat{\varphi}$, try to obtain a more concise variant $\mathfrak{B} \simeq^2_2 \mathfrak{A}$, which realises the same quantifier-free 2-types and realises each $\simeq^2_1$-type of a single element either precisely once or some small finite number of times, by making allocations of quantifier-free 2-types correspondingly.