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Exercises No.12

Exercise 1 [warm-up]
Argue game theoretically that if player **I** has a winning strategy for $\mathsf{G}^k_\infty(\mathfrak{A}, \mathbf{a}; \mathfrak{B}, \mathbf{b})$, then he has a strategy to force a win within $|A|^k \cdot |B|^k$ many rounds.

Exercise 2

(a) With respect to 2-variable equivalence $\equiv^2$ show the following:
  (i) the class of finite linear orderings is closed under 2-variable equivalence $\equiv^2$.
  (ii) two well-orderings (ordinals) are 2-pebble equivalent (w.r.t. G^k_∞) if and only if they are isomorphic. Similarly for any two well-ordered graphs.

(b) Show that, despite (a), the class of finite linear orderings is not definable (as a class of finite $<$-structures) by any sentence of $\mathrm{FO}^2(<)$.

Hint for (b): show that player **II** can win the m-round 2-pebble game played on a sufficiently long linear ordering versus its variant with a single $<$-edge in a suitable position reversed.

Exercise 3
Give examples of pairs of non-isomorphic finite graphs that are indistinguishable in FO^k (k-pebble equivalent), for given levels $k \geqslant 2$.
Can you find examples of such pairs of graphs in which k-pebble equivalence persists even w.r.t. the fragment of FO that has just k variables in the presence of "counting quantifiers" $\exists^{\geqslant i} x_j$ for all $i \geqslant 1$, where $\mathfrak{A}, \mathbf{a} \models \exists^{\geqslant i} x_j \varphi$ iff $|\{a \in A \colon \mathfrak{A}, \mathbf{a}\frac{a}{j} \models \varphi\}| \geqslant i$?
Devise an Ehrenfeucht–Fraïssé game for these more powerful levels of k-pebble equivalence with counting.

Suggested Homework Exercises

Exercise 4
Show that, classically, a first-order formula $\varphi \in \mathrm{FO}_k(\sigma)$ in some relational signature σ is equivalently expressible in $\mathrm{FO}^k(\sigma)$ if, and only if, it is invariant under the equivalence induced by the k-pebble game $\mathsf{G}^k_\infty(\mathfrak{A}, \mathbf{a}; \mathfrak{B}, \mathbf{b})$.
What is the status of this characterisation in finite model theory?

Exercise 5 [Cf. Exercise 9.1]
Show that evenness of the size of a finite set is not definable by a sentence in $\mathrm{MSO}(\emptyset)$. Use this, and a suitable reduction argument, to show (again) that the property of having an even number of atoms is not FO-definable over finite boolean algebras, while it is $<$-invariantly FO-definable.

Hint: instead of the standard format one may use, for *finite* boolean algebras, an alternative two-sorted encoding of a set (the first sort, the set of atoms) together with its power set (the second sort) and with the element relation between the two sorts.

Exercise 6

Over the finite linear orderings $([n], <)$, $[n] = \{1, \ldots, n\}$ with the natural ordering, consider the graph of addition as a ternary relation $R^{[n]} := \{(a, b, c) \in [n]^3 : a + b = c\}$. Show that $R^{[n]}$ is

(a) not uniformly FO-definable over the $([n], <)$;

(b) uniformly implicitly definable in $\mathrm{FO}(\{<, R\})$;

(c) *not* uniformly explicitly definable in $\mathrm{MSO}(<)$.