Classical and Non-Classical Model Theory

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Exercises No.11

Exercise 1 [warmup: expressive power of MSO]
Which of the following queries are MSO definable?

(i) Reachability of a green node (e.g., in directed, finite graphs).
(ii) Acyclicity of directed, finite graphs.
(iii) Mirror symmetry of finite coloured linear orderings.
(iv) Planarity of finite graphs.

Exercise 2 [Deterministic Finita Automata and the Theorem of Myhill-Nerode]
Let Σ be a finite alphabet. A *deterministic finite automaton (DFA)* $\mathcal{A}$ is a tuple (Q, δ, q_0, A) consisting of

(i) a finite set Q (the set of *states* of the automaton),
(ii) a function $\delta\colon Q \times \Sigma \to Q$ (the *transition function*),
(iii) a state $q_0 \in Q$ (the *initial state*), and
(iv) a set $A \subseteq Q$ of *accepting states*.

We define the extended transition function $\delta^*\colon Q \times \Sigma^* \to Q$ by

$$\delta^*(q, \varepsilon) := q,$$
$$\delta^*(q, wa) := \delta(\delta^*(q, w), a)$$

for $q \in Q$, $a \in \Sigma$ and $w \in \Sigma^*$. The automaton is said to *accept* a word w iff $\delta^*(q_0, w) \in A$. The *language accepted by* $\mathcal{A}$ is defined as

$$L_{\mathcal{A}} := \{w \in \Sigma^*\colon \mathcal{A} \text{ accepts } w\}.$$

Let $L \subseteq \Sigma^*$ be a language. Then t.f.a.e.:

(i) L is regular (i.e., $L = L_{\mathcal{A}}$ for some DFA $\mathcal{A}$),
(ii) $\approx_L$ has finite index
(iii) $\sim_L$ has finite index
(iv) L is a union of classes for some finite index right-invariant equivalence relation on Σ^*.

Hint: To show that (iv) implies (i), construct a DFA with $\Sigma^*/\!\sim$ as state space and transition function $([w]_\sim, a) \mapsto [wa]_\sim$. What makes the latter well-defined?

Suggested Homework Exercises

Exercise 3 [MSO-definable subsets of $\mathbb{N}$]
For $S \subseteq \mathbb{N}$ consider the class ORD(S) of those finite linear orderings whose length is in S. Show that ORD(S) is MSO-definable if, and only if, S is *periodic* in the sense of being a finite union of sets of the form $\{a + np\colon n \in \mathbb{N}\}$ for suitable $a, p \in \mathbb{N}$.

Hint: The elements of ORD(S) can be viewed as strings over a unary alphabet. In Exercise 2, we showed that there is DFA $\mathcal{A}$ with ORD$(S) = L_{\mathcal{A}}$. Examine the structure of this DFA.

Exercise 4 [Hanf's Theorem with cut-off]
Let σ be a finite relational signature and $\ell \in \mathbb{N}$. For a σ-structure $\mathfrak{A}$ and $a \in A$, the *ℓ-neighbourhood type* $\iota^{\ell}_{\mathfrak{A}}(a)$ is the isomorphism type of $(\mathfrak{A} \upharpoonright N^{\ell}(a), a)$. For a neighbourhood type ι and structure $\mathfrak{A}$ we set

$$|\mathfrak{A}|_{\iota,\ell} := |\iota(\mathfrak{A})| = |\{a \in A\colon \iota^{\ell}_{\mathfrak{A}}(a) = \iota\}|.$$

Define $\ell(m)$ by

$$\ell(0) := 1$$
$$\ell(m+1) := 3 \cdot \ell(m) + 1,$$

i.e., $\ell(m) = (3^{m+1} - 1)/2$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two σ-structures and $m \in \mathbb{N}$. Suppose that for some $e \in \mathbb{N}$, the $\ell(m)$-neighbourhoods in $\mathfrak{A}$ and $\mathfrak{B}$ have less than e elements, and that for each $\ell(m)$-neighbourhood type ι

$$|\mathfrak{A}|_{\iota,\ell(m)} = |\mathfrak{B}|_{\iota,\ell(m)} \quad \text{or} \quad |\mathfrak{A}|_{\iota,\ell(m)}, |\mathfrak{B}|_{\iota,\ell(m)} \geqslant e \cdot m.$$

Show that $\mathfrak{A} \equiv_m \mathfrak{B}$.

Hint: Show that $(I_j)_{j \leqslant m}$ defined by

$$I_j := \Big\{\bar{a} \mapsto \bar{b}\colon \Big(\mathfrak{A} \upharpoonright \bigcup N^{\ell(j)}(a_i), \bar{a}\Big) \simeq \Big(\mathfrak{B} \upharpoonright \bigcup N^{\ell(j)}(b_i), \bar{b}\Big) \text{ and length}(\bar{a}) \leqslant m - j\Big\}$$

defines a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, with the convention that

$$I_m = \{() \mapsto ()\}$$

contains the single mapping which maps the empty tuple to the empty tuple. When extending a partial mapping $\bar{a} \mapsto \bar{b} \in I_j$, distinguish two cases according to whether the $\ell(m-j)$-neighbourhood of the newly chosen element touches the $\ell(m-j)$-neighbourhood of some previously chosen element or not.