Classical and Non-Classical Model Theory

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Exercises No.10

Exercise 1 [warm-up: Gaifman equivalents]
Let σ consist of a binary edge relation E and a unary predicate P. Give first-order formalisations in Gaifman normal form (boolean combinations of local formulae and basic local sentences) for the following:

(i) $\varphi_1(x) := \exists y(x \neq y \wedge Py)$
(ii) $\varphi_2(x) := \exists y(x \neq y \wedge \neg Exy \wedge Py)$

Exercise 2 [minimal models]
Let σ be finite and relational. Recall the weak substructure relationship $\mathfrak{A} \subseteq_w \mathfrak{B}$ between σ-structures, meaning that $A \subseteq B$ and $R^{\mathfrak{A}} \subseteq R^{\mathfrak{B}}$ for every $R \in \sigma$. Let $\mathcal{C} \subseteq \mathsf{Fin}(\sigma)$ be closed under homomorphisms within $\mathsf{Fin}(\sigma)$. Then the following are equivalent:

(i) $\mathcal{C} = \mathsf{FMod}(\varphi)$ for some existential positive FO(τ)-sentence φ.
(ii) $\mathcal{C}$ has finitely many $\subseteq_w$-minimal members, up to isomorphism.
(iii) $\mathcal{C}$ has finitely many $\subseteq$-minimal members, up to isomorphism.

Exercise 3 [wideness]
A class $\mathcal{C}$ of (finite) relational structures is called *wide*, if there is a function $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that, for all ℓ, m, the Gaifman graph of every structure $\mathfrak{A} \in \mathcal{C}$ contains an ℓ-scattered m-tuple provided $|A| \geqslant f(\ell, m)$. Show that the class of all finite graphs of degree up to d is wide, for any d.

Hint: consider a case distinction as to many connected components vs. components of large diameter.

Suggested Homework Exercises

Exercise 4 [extra: universal algebra of cores]
Two relational structures are *homomorphically equivalent* if there are homomorphisms between them in both directions. A *core* is a structure that is not homomorphically equivalent to any of its proper weak substructures.
Show the following, for any finite relational σ:

(i) every $\mathfrak{A} \in \mathsf{Fin}(\sigma)$ possesses a core in the sense that $\mathfrak{A}$ is homomorphically equivalent to some core $\mathfrak{A}_0 \subseteq_w \mathfrak{A}$.
(ii) any two cores of any two homomorphically equivalent finite structures are isomorphic. In particular, the core of any given $\mathfrak{A} \in \mathsf{Fin}(\sigma)$ is unique up to isomorphism.
(iii) any core $\mathfrak{A}_0 \subseteq_w \mathfrak{A}$, is related to $\mathfrak{A}$ by a *retract*, i.e., by a homomorphism whose restriction to $A_0 \subseteq A$ is the identity: $h\colon \mathfrak{A} \xrightarrow{\text{hom}} \mathfrak{A}_0$ with $h\restriction A_0 = \mathrm{id}_{A_0}$.

Exercise 5 [one restricted version of Lyndon–Tarski in FMT]
Let $\mathcal{C} \subseteq \mathsf{Fin}(\sigma)$ for finite relational σ be closed under substructures and disjoint unions within $\mathsf{Fin}(\sigma)$, as well as wide (see Exercise 3 above). Assume that, like any FO-definable class, $\mathcal{C}$ is also closed under (ℓ, q, m)-Gaifman-equivalence $\equiv^{\ell}_{q,m}$ within $\mathsf{Fin}(\sigma)$ (for suitable ℓ, q, m).

(a) Let $\mathfrak{A} \Rightarrow^{\ell}_{q,1} \mathfrak{B}$ be the transfer relationship saying that for all $\psi(x) \in \mathrm{FO}_1(\sigma)$ of quantifier rank $\mathrm{qr}(\psi) \leqslant q$, $\mathfrak{A} \models \exists x \psi^{\ell}(x)$ implies $\mathfrak{B} \models \exists x \psi^{\ell}(x)$. Show that for L, Q that are sufficiently large (in relation to ℓ, q) the following holds for all $\mathfrak{A} \in \mathcal{C}$ and $a, b \in A$ distance $d(a,b) > 2L$:

$$\mathfrak{A}{\restriction} N^L(a), a \equiv_Q \mathfrak{A}{\restriction} N^L(b), b \quad \Rightarrow \quad \mathfrak{A} \Rightarrow^{\ell}_{q,1} \mathfrak{B} := \mathfrak{A}{\restriction}(A \setminus \{b\}).$$

(b) Show that for N that is sufficiently large (in relation to L, Q and the wideness bounds on $\mathcal{C}$), any $\mathfrak{A} \in \mathcal{C}$ of size $|A| > N$ must have elements $a, b \in A$ distance $d(a,b) > 2L$ such that $\mathfrak{A}{\restriction} N^L(a), a \equiv_Q \mathfrak{A}{\restriction} N^L(b), b$.

(c) Conclude that, if $\mathcal{C}$ is also closed under homomorphisms within $\mathsf{Fin}(\sigma)$, it cannot have any $\subseteq$-minimal (or $\subseteq_w$-minimal) members of size greater than N.

Hint for (c): in the situation of part (a), the disjoint union of $\mathfrak{A}$ with m copies of $\mathfrak{B}$ will be $\equiv^{\ell}_{q,m}$-equivalent to the disjoint union of just m copies of $\mathfrak{B}$ (why?). And $\mathfrak{A}$ admits a homomorphism into the former while the latter admits a homomorphism into $\mathfrak{B}$.

Remark: an FMT version of the Lyndon–Tarski correspondence between preservation under homomorphisms and positive existential definability obtains in restriction to wide classes of finite relational structures that are closed under substructures and under disjoint unions.[1] You may want to piece this together from the above.

[1] A corresponding result of Atserias–Dawar–Kolaitis also works over other classes, and is independent of the full FMT analogue of Lyndon–Tarski proved by Rossman.