Classical and Non-Classical Model Theory

Martin Otto
Kord Eickmeyer
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Exercises No.1

Exercise 1 [filters]
Consider $I \neq \emptyset$ and $\mathcal{B} \subseteq \mathcal{P}(I)$ with the *finite intersection property* (*f.i.p.*). Show the following:

(a) closure under intersection ($\cap$) and supersets ($\supseteq$) preserves f.i.p.

(b) $\mathcal{F}(\mathcal{B})$, defined as the closure under supersets of the closure under (finite) intersections of $\mathcal{B}$, is a filter extending $\mathcal{B}$.

(c) if $s, \overline{s} \notin \mathcal{B}$ then at least one of $\mathcal{B} \cup \{s\}$ or $\mathcal{B} \cup \{\overline{s}\}$ has f.i.p. (and hence extends to a filter).

(d) if $\mathcal{B} \subseteq \mathcal{P}(I)$ is *maximal* with f.i.p., then $\mathcal{B} = \mathcal{F}(\mathcal{B})$ is an ultrafilter on I.

Use Zorn's Lemma to show that every $\mathcal{B} \subseteq \mathcal{P}(I)$ with f.i.p. can be extended to an ultrafilter $\mathcal{U} \supseteq \mathcal{B}$.

Exercise 2 [cf. Lemma 1.3]
Let $\mathcal{F}$ be a filter on I.

(a) Show that the relation $\sim_{\mathcal{F}}$ is an equivalence relation on $\prod_i A_i$ (for any family of non-empty sets $(A_i)_{i \in I}$).

(b) Show that $\sim_{\mathcal{F}}$ is a congruence w.r.t. any function $f^{\mathfrak{A}}$ in the direct product $\mathfrak{A} := \prod_i \mathfrak{A}_i$ of a family of structures $(\mathfrak{A}_i = (A_i, f^{\mathfrak{A}_i}))_{i \in I}$.

(c) Show that for a family of structures $(\mathfrak{A}_i = (A_i, R^{\mathfrak{A}_i}))_{i \in I}$ with relation R, and for $\mathbf{a} \sim_{\mathcal{F}} \mathbf{a}'$ (component-wise equivalence): $\|R\mathbf{a}\| \in \mathcal{F} \Leftrightarrow \|R\mathbf{a}'\| \in \mathcal{F}$.

Exercise 3 [cf. Łos Theorem, Theorem 1.5]
Which steps in the inductive treatment of the usual connectives and quantifiers of FO go through for arbitrary filters (not necessarily ultrafilters)? Provide proofs or counterexamples (for $\vee$, $\forall$, and $\neg$).

Suggested Homework Exercises

Exercise 4 [reduced products preserve Horn formulae]
A *Horn clause* is a formula of the form

$$\varphi = (\theta_1 \wedge \ldots \wedge \theta_m) \to \sigma,$$

where $\theta_1, \ldots, \theta_m$ and σ are atomic formulae. We allow $m = 0$ and $\sigma = \bot$. A *Horn formula* is a formula that is built from Horn clauses using $\wedge$, $\exists$ and $\forall$.

Show that Horn formulae are preserved under taking reduced products, i.e., if φ is a Horn formula and $(\mathfrak{A}_i, \mathbf{a}(i))$ such that

$$\|\varphi[\mathbf{a}]\| \in \mathcal{F},$$

then

$$\mathfrak{A}^I/\mathcal{F} \models \varphi.$$

Exercise 5 [non-standard models]
Analyse non-standard extensions $\mathfrak{R}^* \succcurlyeq \mathfrak{R}$ and $\mathfrak{N}^* \succcurlyeq \mathfrak{N}$ obtained as ultrapowers $\mathfrak{A}^{\mathbb{N}}/\mathcal{U}$ w.r.t. suitable ultrafilters $\mathcal{U}$ over $\mathbb{N}$ in relation to the base structures $\mathfrak{A} = \mathfrak{R}, \mathfrak{N}$.
For instance,

(i) which sequences $(a_n) \in \mathbb{N}^{\mathbb{N}}$ represent infinitely large numbers?

(ii) which sequences $(a_n) \in \mathbb{R}^{\mathbb{N}}$ represent points in the infinitesimal neighbourhood of $a \in \mathbb{R}$?

(iii) what are the possible order-types for $\mathfrak{N}^*$ and $\mathfrak{R}^*$?

(iv) investigate the equivalence relations of being "finitely far apart" in $\mathfrak{N}^*$ and of being "of the same order of magnitude" in $\mathfrak{R}^* \setminus \{0\}$.