Algorithmic Discrete Mathematics
6. Exercise Sheet

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Groupwork

Exercise G1

We consider an application of Hall's Marriage Theorem:

Let $X = \{S_i \mid 1 \leq i \leq k\}$ be a finite family of sets. A *system of distinct representatives (SDR)* or *transversal* of X is a set T with a bijection $\varphi : T \to X$ such that $t \in \varphi(t)$.

Prove the following theorem:

X has an SDR if and only if for any $j \in \{1, \ldots, k\}$ the union of any j of the S_i has size at least j.

Exercise G2

For each of the following families of sets determine whether the condition of the theorem on SDRs is met. If so, then find an SDR and the corresponding bijection φ. If not, then show how the condition is violated.

(a) $\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5\},\{1,2,5\}$

(b) $\{1,2,4\},\{2,4\},\{2,3\},\{1,2,3\}$

(c) $\{1,2\},\{2,3\},\{1,2,3\},\{2,3,4\},\{1,3\},\{3,4\}$

(d) $\{1,2,5\},\{1,5\},\{1,2\},\{2,5\}$

(e) $\{1,2,3\},\{1,2\},\{1,3\},\{1,2,3,4,5\},\{2,3\}$

Exercise G3

Consider the following problem: Assume that there are n factories, producing a supply of $s_1, \ldots, s_n$ of some good. Moreover, there are m customers, each asking for a demand of $d_1, \ldots, d_m$. Each factory i can deliver an amount $c_{ij} \geq 0$ to the customer j.

(a) Model the problem as a flow problem.

(b) Now consider the following real world problem: There are six universities that will produce five mathematics graduates each. Moreover, there are five companies that will be hiring $7, 7, 6, 6, 5$ math graduates, respectively. No company will hire more than one student from any given university. Will everyone get a job?

Exercise G4

We want to construct an $(n \times m)$-matrix whose entries are nonnegative integers such that the sum of the entries in row i is r_i and the sum of the entries in column j is c_j. (Then clearly $\sum r_i = \sum c_j$.)

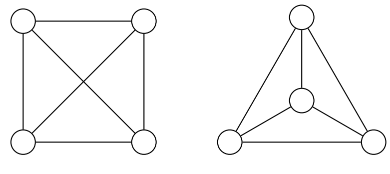
(a) What other constraints (if any) should be imposed on the r_i and c_j to assure such a matrix exists?

(b) Construct such a (5×6)-matrix with row sums $20, 40, 10, 13, 25$ and column sums all equal to 18.

Exercise G5

A graph is *planar* if it can be drawn (or *embedded*) in the plane without intersecting edges.

For example, consider the graph K_4. The first drawing is not planar, the second one is:



So K_4 is a planar graph.

An embedding of a planar graph subdivides $\mathbb{R}^2$ into connected components, the *faces*. *E.g.* the planar drawing of K_4 above has four faces, three bounded triangular ones and one unbounded face.

Prove *Euler's Formula* for a connected planar graph $G = (V, E)$ with $|V|$ vertices, $|E|$ edges and $F(G)$ faces:

$$|V| - |E| + F(G) = 2.$$

Homework

Exercise H1 (5 points)
Determine the number of perfect matchings in

(a) $K_{n,n}$ and

(b) K_{2n}.

Exercise H2 (5 points)
Let $G = (A \cup B, E)$ be a d-regular bipartite graph for $1 \leq d$, *i.e.,* all vertices have degree d.

(a) Show that $|A| = |B|$.

(b) Show that G contains a perfect matching.

(c) Using induction, prove that the edges of G can be partitioned into d perfect matchings.

Exercise H3 (5 points)
A *(minimal) node cover* of a graph $G = (V, E)$ is a subset $U \subseteq V$ (with minimal cardinality) such that for every $\{v, w\} \in E$ at least one of v, w is contained in U, *i.e.,* $\{v, w\} \cap U \neq \emptyset$).

(a) Prove the *Theorem of König*:
In a bipartite graph the size ν of a maximal matching equals the size τ of a minimal node cover.

(b) Conclude that a bipartite graph has a perfect matching if and only if every node cover has size at least $\frac{1}{2}|V|$.

Exercise H4 (5 points)

(a) Show that every graph with a least six vertices contains the graph K_3 or its complement $\overline{K_3}$.

(b) Show that every graph with a least ten vertices contains K_4 or $\overline{K_3}$.

(c) Show that the assertion in (b) does not hold for graphs with eight vertices.