# Algorithmic Discrete Mathematics
# 5. Exercise Sheet

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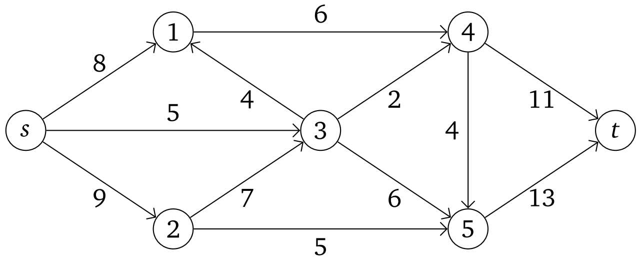
Groupwork

**Exercise G1**

Using the Ford-Fulkerson method, compute a maximal flow in the following network:

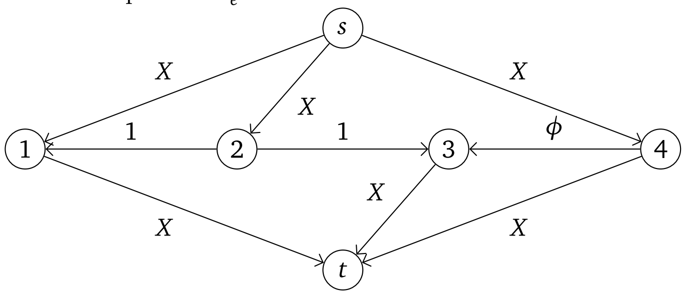



Also determine a minimal cut in $G$.

**Exercise G2**

The goal of this exercise is to show that the Ford-Fulkerson method need not terminate if we allow irrational edge capacities.

Consider the following network with capacities $c_e$.




Here $X$ is some large integral constant and $\phi = \frac{1}{2}(\sqrt{5}-1)$. (Note that $\phi^n = \phi^{n+1} + \phi^{n+2}$ for any $n \geq 0$.)

(a) Show by induction that for any integer $n \geq 0$ the residual capacities of the three horizontal edges can be brought to the values $\phi^n, 0, \phi^{n+1}$.

(b) Conclude that Ford-Fulkerson need not terminate on this network. Does it converge?

(c) Find a network where Ford-Fulkerson converges, but not to a maximal flow.

**Exercise G3**

Let $(G = (V, E), s, t, c)$ be a network with integral capacities $c(e) \in \mathbb{Z}$ for all edges $e \in E$. Prove or refute the following assertions:

(a) If all capacities are even then there is a maximal $(s-t)$-flow $f$ such that $f(e)$ is even for all $e \in E$.

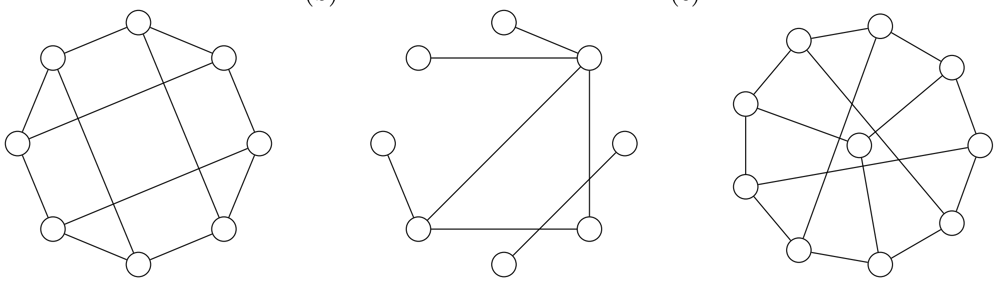
(b) If all capacities are odd then there is a maximal $(s-t)$-flow $f$ such that $f(e)$ is odd for all $e \in E$.

**Exercise G4**

Let $G = (V, E)$ be a graph. A subset $M \subseteq E$ is a *matching* in $G$ if $m \cap m' = \emptyset$ for all $m, m' \in M$. A matching $M$ is *perfect* if $2|M| = |V|$.

In each of the following graphs determine a perfect matching or show that no perfect matching exists.

(a) (b) (c)



## Homework

**Exercise H1** (5 points)
Let $(G=(V,E),s,t,c)$ be a network with integral capacities $c(e)\in\mathbb{Z}_+$ for all $e\in E$. Let $f$ be a maximal flow in this network. We assume that the capacity of one edge $e$

(a) is increased by 1,
(b) is decreased by 1.

Describe an algorithm with complexity $\mathcal{O}(m+n)$ that determines a maximal flow in the new network. Improve your algorithm (or your analysis) to $\mathcal{O}(m)$.

**Exercise H2** (5 points)

(a) An edge $e$ in a network $(G=(V,E),s,t,c)$ where $t$ can be reached from $s$, is called *upwards critical* if increasing the capacity of $e$ increases the value of the maximal flow. Does every network possess an upwards critical edge?
Describe an algorithm that finds all upwards critical edges and has a considerably better running time than solving $m$ max flow problems.

(b) An edge $e$ in a network $(G=(V,E),s,t,c)$ where $t$ can be reached from $s$, is called *downwards critical* if decreasing the capacity of $e$ decreases the value of the maximal flow. Does every network possess a downwards critical edge?
Describe an algorithm that finds all downwards critical edges and analyse its running time.

**Exercise H3** (5 points)
Let $G=(V,E)$ be an undirected graph and $s\neq t\in V$. A subset $F\subseteq E$ is $(s-t)$-*separating* if any $(s-t)$-path uses at least one edge of $F$. A collection $P_1,\dots,P_k$ of $(s-t)$-paths in $G$ is *edge-disjoint* if no pair $P_i,P_j,i\neq j$ have an edge in common.

(a) Prove the *edge version of the Theorem of Menger*:
The maximal number of edge-disjoint paths in $G$ equals the minimal size of an $(s-t)$-separating edge set.
*Hint:* Apply the MaxFlow-MinCut Theorem to a suitable network.

(b*) (Bonus exercise – no points) A subset $U\subseteq V$ is $(s-t)$-*separating* if any $(s-t)$-path uses at least one node of $U$. Two $(s-t)$-paths are *internally disjoint* if they only share the nodes $s$ and $t$.
Prove the *node version of the Theorem of Menger*:
Assume $\{s,t\}\notin E$. Then the maximal number of internally disjoint $(s-t)$-paths equals the minimal size of an $(s-t)$-separating node set.
*Hint:* Construct a directed graph as above, then replace each node $v\in V\setminus\{s,t\}$ by a pair $v^-,v^+$ and a directed edge $(v^-,v^+)$. Again apply the MaxFlow-MinCut Theorem.

**Exercise H4** (5 points)
Let $G=(V,E)$ be a directed graph, $c:E\to\mathbb{R}_+$ a capacity and $f:E\to\mathbb{R}$ a flow on $G$. Prove or disprove the following statements:

(a) $f$ is maximal $\Rightarrow f(e)=0$ or $f(e)=c(e)$ for all $e\in E$.
(b) There is a maximal flow such that $f(e)=0$ or $f(e)=c(e)$ for all $e\in E$.
(c) A minimal cut is unique if all capacities are pairwise distinct.
(d) Multiplying all capacities $c(e)$ by a number $\lambda>0$ does not change the minimal cuts.
(e) Adding a number $\lambda>0$ to all capacities $c(e)$ does not change the minimal cuts.