Algorithmic Discrete Mathematics
4. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

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SS 2013
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Due to the holiday on Thursday, this week's exercise will take place on

Friday, 13:30 in S1|03 123.

In addition, the Wednesday group will meet as usual. But participants of the Wednesday group are encouraged to also come on Friday since there will be a short discussion on Prim's algorithm in the beginning.

Groupwork

Exercise G1

Prim's algorithm is another algorithm to compute a minimal spanning tree. It starts with some root node r and then successively builds up a tree by adding a minimal edge connected to the partial tree constructed so far.

The algorithm will be presented in the exercise on 31 May 2013.

The pseudocode is as follows:

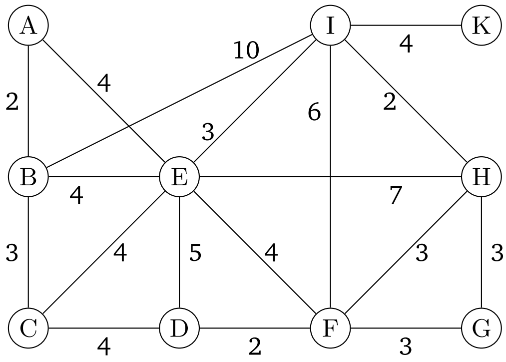
Algorithm 1: Prim's Algorithm

Input: connected graph $G = (V, E)$ given as adjacency list, weight function $w : E \to \mathbb{R}$, root node $r \in V$

Output: minimal spanning tree $T = (V, \{(v, \text{pred}(v)) \mid v \in V \setminus \{r\}\})$ of G

```
foreach v ∈ V do
    pred(v) ← 0
    dist(v) ← ∞ // distance from tree
Q ← V // priority queue
dist(r) ← 0
while Q ≠ ∅ do
    v ← extract_min(Q) // vertex with minimal distance
    foreach u ∈ Adj(v) do
        if u ∈ Q and w(u,v) < dist(u) then
            pred(u) ← v
            dist(u) ← w(u,v)
```

(a) Perform Prim's algorithm on the following graph (starting at node A):



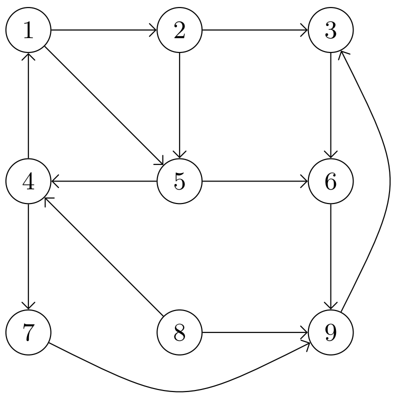
(b) Prove that Prim's Algorithm correctly computes a minimal spanning tree.

Exercise G2

A directed graph is called *strongly connected* if there is a path from each vertex to every other vertex. It is called *weakly connected* if the underlying undirected graph is connected.

The *strongly connected components (strong components)* of a directed graph are its maximal strongly connected subgraphs.

(a) Determine the strong components of the following graph.



(b) Can you redirect some of the edges so that the graph becomes strongly connected?

(c) Show that a directed graph is *acyclic*, *i.e.*, it does not contain any directed cycle, if and only if all strong components consist of only one vertex. (Note that cycles in directed graphs may consists of only two nodes.)

Exercise G3

A *bridge* in an undirected graph $G = (V, E)$ is an edge e such that $G - e$ has more connected components than G. Prove that a graph in which every vertex has even degree does not have a bridge.

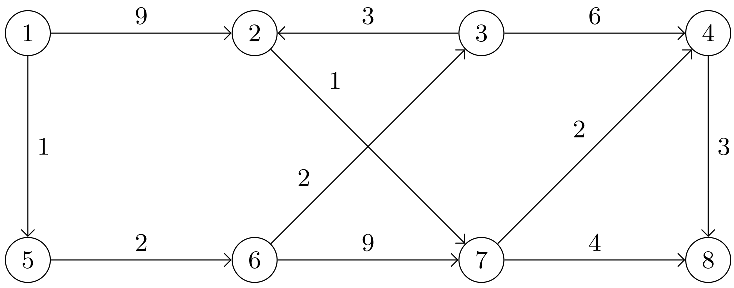
Exercise G4

Two graphs are called *isomorphic* ($G \cong H$) if there is a bijective map between their vertex sets that preserves adjacency. Recall that the complementary graph $\overline{G}$ of a graph $G = (V, E)$ is defined as $\overline{G} = (V, \binom{V}{2} \setminus E)$. If $G \cong \overline{G}$ then we say that G is *self-complementary*. Prove: Every self-complementary graph has $4k$ or $4k + 1$ vertices for some $k \in \mathbb{N}$.

Homework

Exercise H1 (5 points)

(a) Consider the following graph:



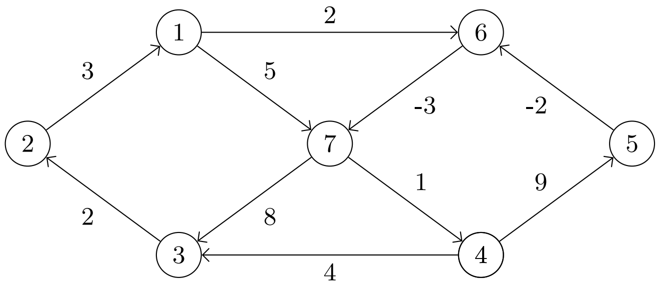
Using Dijkstra's Algorithm compute a shortest path from $r = 1$ to all other nodes and determine the shortest path tree.

(b) Is the shortest path tree for any given graph and root node unique?

(c) Show by an example that Dijsktra's Algorithm does not work correctly for negative edge weights.

Exercise H2 (5 points)

(a) Consider the following graph:



Using the Algorithm of Bellman-Ford compute a shortest path from $r = 1$ to all other nodes. In each step specify the order in which you process the edges. Also draw the shortest path tree.

(b) Is the shortest path tree of a graph a minimal spanning tree (of the underlying undirected graph)?

Exercise H3 (5 points)

Let $G = (V, E)$ be a connected undirected graph and $T = (V, E(T))$ a spanning tree of G. A *swap* is a pair (e, f) of edges with $e \in E(T), f \notin E(T)$ such that $T' = T - e + f$ is a spanning tree.

(a) Can any spanning tree of G be transformed into any other spanning tree via a finite sequence of swaps?

(b) What is the maximal number of swaps needed for this?

Exercise H4 (5 points)

(a) Let G be a graph with $n \geq 2$ vertices and $m > \binom{n-1}{2}$ edges. Show that G is connected.

(b) Show that (up to isomorphisms) there is only one disconnected graph with n vertices and $\binom{n-1}{2}$ edges.