Algorithmic Discrete Mathematics
2. Exercise Sheet

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Groupwork

Exercise G1

(a) Show that every tree T has at least $\Delta(T)$ leaves.

(b) Show that a tree without a vertex of degree 2 has more leaves than other vertices.

Exercise G2

Recall that an automorphism of a graph $G = (V, E)$ is a bijective map $\phi : V \to V$ such that $\{u, v\} \in E \Leftrightarrow \{\phi(u), \phi(v)\} \in E$ for any $u, v \in V$.
Show that every automorphism of a tree fixes a vertex or an edge.

Exercise G3

Let $f, g : \mathbb{N} \to \mathbb{N}$ be two functions such that $g \in \Omega(f)$. Assume that two algorithms are given:

- Algorithm A has a running time of $\mathcal{O}(f)$.
- Algorithm B has a running time of $\mathcal{O}(g)$.

Consider the following two algorithms:

Algorithm 1:

Input: $n \in \mathbb{N}$
$s \leftarrow 0$
for $i = 0, \ldots, 100$ **do**
    Run Algorithm A
for $i = 0, \ldots, 2n$ **do**
    Run Algorithm B

Algorithm 2:

Input: $n \in \mathbb{N}$
if $n \geq 100$ **then**
    Run Algorithm A
else
    Run Algorithm B

Estimate the running times as accurately as possible.

Exercise G4

Let $f, g : \mathbb{N} \to \mathbb{N}$ be two functions and a a constant. Prove:

(a) $f \in \mathcal{O}(f)$

(b) $a \cdot \mathcal{O}(f) \subseteq \mathcal{O}(f)$

(c) $\mathcal{O}(f) + \mathcal{O}(f) \subseteq \mathcal{O}(f)$

(d) $\mathcal{O}(f) \cdot \mathcal{O}(g) \subseteq \mathcal{O}(fg)$

(e) $f \cdot \mathcal{O}(g) \subseteq \mathcal{O}(fg)$

(f) $\max(f, g) \in \Theta(f + g)$

Hint: For two sets A, B addition and multiplication are defined point-wise, *e.g.* for $A = \{a, b\}$ and $B = \{c, d\}$: $A + B = \{a + c, a + d, b + c, b + d\}$, $A \cdot B = \{ac, ad, bc, bd\}$.

Exercise H1 (5 points)
Recall that a *subgraph* of a graph $G=(V,E)$ is a graph $H=(W,F)$ with $W \subseteq V, F \subseteq E$. H is *induced* if $F = E \cap \binom{W}{2}$. Assume that a graph $G=(V,E)$ with $|V|=n$ and $|E| \geq 3$ without isolated vertices does not have an induced subgraph with two edges. Show that $G = K_n, n \geq 3$, *i.e.*, G is a complete graph on n vertices.

Exercise H2 (5 points)
Sort the functions

$$n, \quad n^3, \quad \sqrt{n}, \quad n!, \quad 2^n, \quad n^n$$

according to their complexity in ascending order using o-notation.

Reminder:

$$f \in o(g) \iff \forall c > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \geq n_0 : 0 \leq f(n) < cg(n)$$

Exercise H3 (5 points)
Consider the following algorithm:

Algorithm 3:

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Input: n ∈ ℕ
d ← 2
q ← n
while q > d do
    q ← n/d
    if ⌈q⌉ = q then
        return d
    else
        d ← d + 1
return 0
```

What does it do? Estimate its running time.

Exercise H4 (5 points)
Let $G=(V,E)$ be a connected Eulerian graph. Devise an algorithm that returns an Eulerian tour in G, prove its correctness and estimate its running time in $\mathcal{O}$-notation.