Algorithmic Discrete Mathematics
7. Exercise Sheet

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Groupwork

Exercise G1
You are given twelve coins, eleven of which have identical weight, one is lighter or heavier. How many weighings with a binary scale do you need to find the deviating coin (and the sign of the deviation)? Draw the decision tree and prove that your number of weighings is optimal.

Extra puzzle: Find a solution without case distinctions.

Solution: See next page for a decision tree.

It is not possible to come out with two weighings since the decision tree has $q = 3$ and at least $n = 24$ leaves. Thus, its length is at least $\lceil \log_3 24 \rceil = 3$.

A solution without case distinctions could look like this:

Perform three weighings according to the following table

i	left	right	w_i
1	2,4,10,11	1,5,7,8	1
2	3,4,6,7	2,5,11,12	3
3	5,8,9,10	6,7,11,12	9

and determine $s_i \in \{\pm 1, 0\}$, $i = 1, 2, 3$ for the i-th weighing according to the following rule:

$$s_i = \begin{cases} +1 & \text{if left heavier,} \\ -1 & \text{if right heavier,} \\ 0 & \text{if equal.} \end{cases}$$

Compute $K := \sum_{i=1}^{3} s_i w_i$. Then the weight of the coin $|K|$ deviates.

For $K \in \{1, 2, -3, -4, -5, 6, 7, -8, -9, -10, 11, 12\}$ the coin is heavier. For $K \in \{-1, -2, 3, 4, 5, -6, -7, 8, 9, 10, -11, -12\}$ it is lighter.

Yet another extra puzzle: Why does this work?

Exercise G2
Let $n \geq 1, q \geq 2, \ell_1, \ldots, \ell_n \in \mathbb{Z}_{\geq 0}$. Prove:

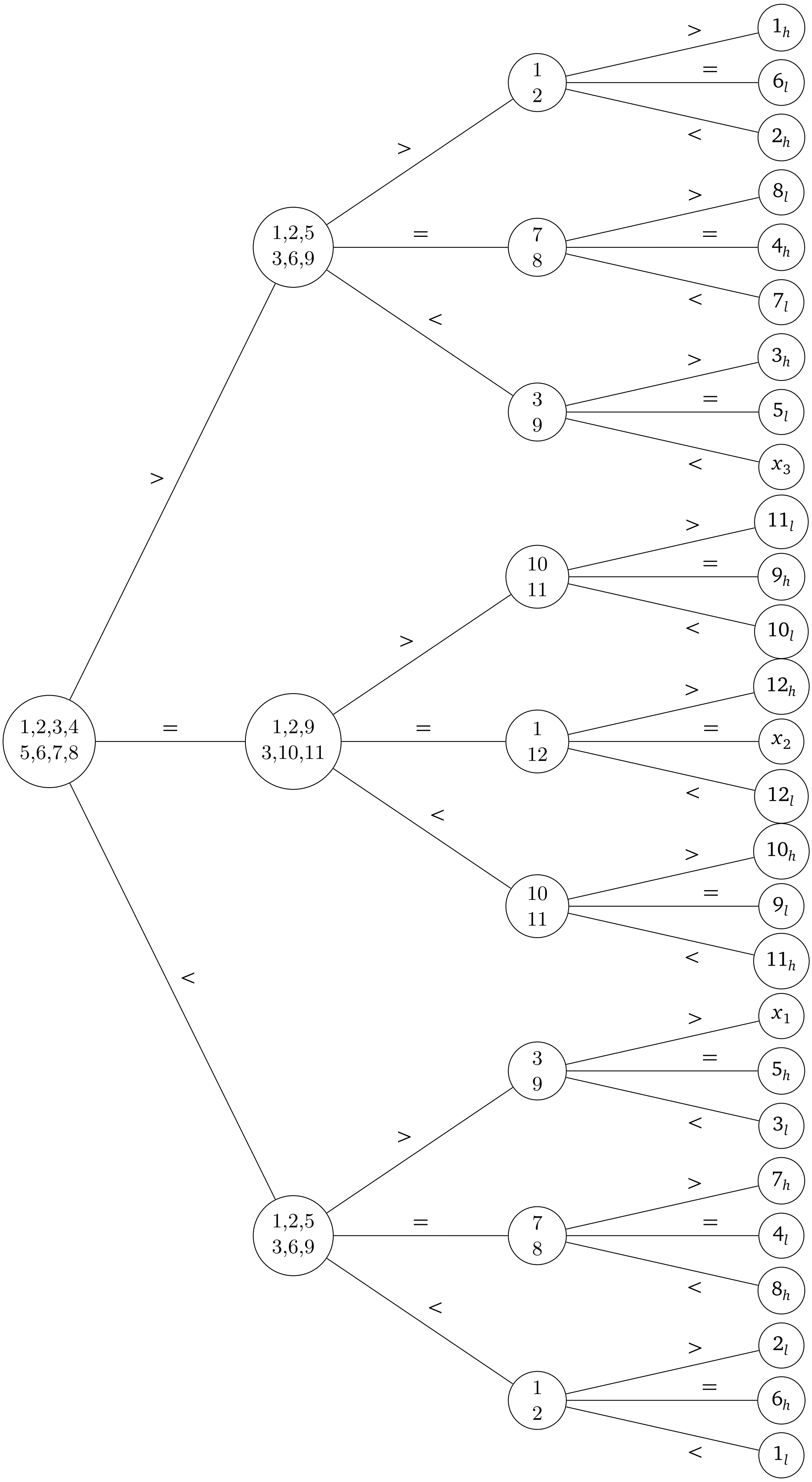
(a) If $T \in T(n,q)$ has leaves of lengths $\ell_1, \ldots, \ell_n$ then $\sum_{i=1}^{n} q^{-\ell_i} \leq 1$. Equality occurs if and only if T is complete.

(b) Given lengths $\ell_1, \ldots, \ell_n$ such that $\sum_{i=1}^{n} q^{-\ell_i} \leq 1$, there is $T \in T(n,q)$ with these lengths.

Solution:

(a) It suffices to consider complete trees. We prove the claim by induction over n. For $n = 0$ the tree consists only of the root and we have $1 = q^0$. So let $n \geq 1$. Construct a new tree T' by replacing some "fork" of leaves of length ℓ by a single leaf of length $\ell - 1$. Then T' is a complete $(n - q + 1, q)$-tree. By induction we get

$$\sum_{i=1}^{n} q^{-\ell_i} = \sum_{i=q+1}^{n} q^{-\ell_i} + q \cdot q^{-\ell} = \sum_{i=q+1}^{n} q^{-\ell_i} + q^{-(\ell-1)} = 1.$$


1,2,3,4
5,6,7,8
1,2,5
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1_h
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8_l
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7_l
3_h
5_l
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10_l
12_h
x_2
12_l
10_h
9_l
11_h
x_1
5_h
3_l
7_h
4_l
8_h
2_l
6_h
1_l

(b) Conversely, assume that $\sum_{i=1}^{n} q^{-\ell_i} \leq 1$. Let $w_k = |\{i \mid \ell_i = k\}|$ for $k = 1, \ldots, L = L(T)$. *I.e.*, w_k is the number of leaves of length k in the tree T that we are going to construct. We can then write the inequality $\sum_{i=1}^{n} q^{-\ell_i} \leq 1$ as

$$\sum_{k=0}^{L} w_k q^{-k} \leq 1$$

or equivalently

$$w_0 q^L + w_1 q^{L-1} + \ldots + w_{L-1} q + w_L \leq q^L.$$

We construct the tree inductively. If $w_0 = 1$ we have $n = 0$ and T consists only of the root. Assume we already constructed T up to length k. *I.e.*, we already constructed $w_0, w_1, \ldots, w_k$ leaves of lengths $0, 1, \ldots, k$, respectively. So we cannot attach any more leaves to those $w_0 + w_1 + \ldots + w_k$ leaves. Hence there are

$$q^k - \sum_{i=0}^{k} w_i q^{k-i}$$

inner nodes at level k.

By the above inequality we have

$$w_{k+1} q^{L-k-1} \leq q^L - \sum_{i=0}^{k} w_i q^{L-i}$$

and hence

$$w_{k+1} \leq q^{k+1} - \sum_{i=0}^{k} w_i q^{k+1-i} = q\left(q^k - \sum_{i=0}^{k} w_i q^{k-i}\right).$$

Thus, we can add all w_{k+1} leaves of length $k+1$.

Exercise G3

Assume you want to encode the symbols $a, \ldots, z$ over the alphabet $A = \{0, 1, 2\}$.

(a) Devise an equal length code (*i.e.*, every symbol is encoded with a code word of equal length) for this problem and encode the following sentence using your code (without punctuation and case sensitivity):

The lecturer's name is Andreas Paffenholz.

(b) Now construct an optimal prefix code for the above sentence using Huffman's Algorithm by counting the occurrences of the letters, encode the sentence again and compare the length of the output with the equal length code above.

Solution:

(a) We need code words of length 3. Thus the encoded string has length $3 \cdot 35 = 105$.

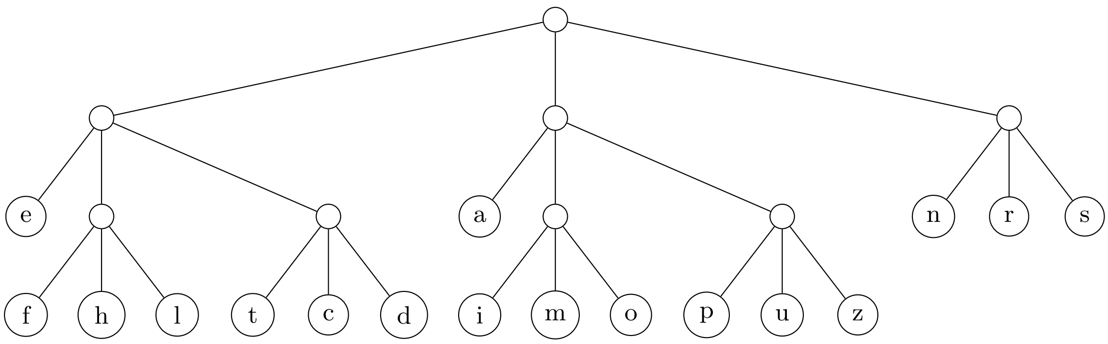
(b) The weights are:

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
4	0	1	1	6	2	0	2	1	0	0	2	1	3	1	1	0	3	3	2	1	0	0	0	0	1

We sort them increasingly. If we ignore the ones with weight 0 we have $n = 17, q = 3$.

e	a	n	r	s	f	h	l	t	c	d	i	m	o	p	u	z	b	g	j	k	q	v	w	x	y
6	4	3	3	3	2	2	2	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0

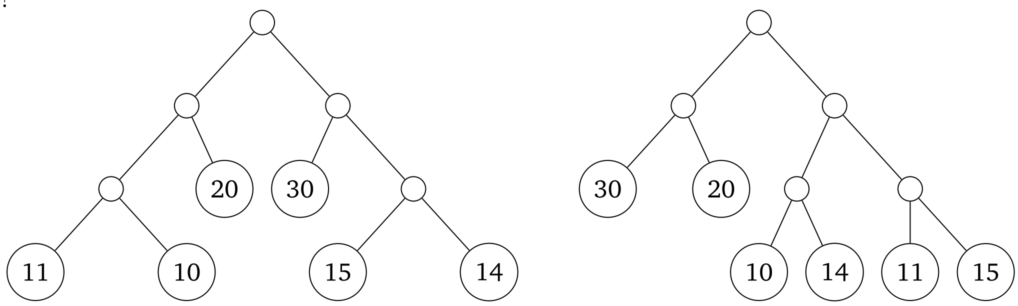
We get the following tree T:



The length of the output equals $\overline{L}(T) = 86$.

Exercise G4

Given $p_1, \ldots, p_6 = 10, 11, 14, 15, 20, 30$ show that the following decision trees are both optimal. Which one is also a Huffman tree?



Solution: The first one is a Huffman tree and hence optimal. The second one has the same weight but is no Huffman tree.

Exercise G5

Consider the following game for two players: Start with a natural number $n > 1$. Each player takes turns to divide the current number by a power of a prime number (different from 1). If the current number is 1 at the beginning of one player's round, then that player loses.

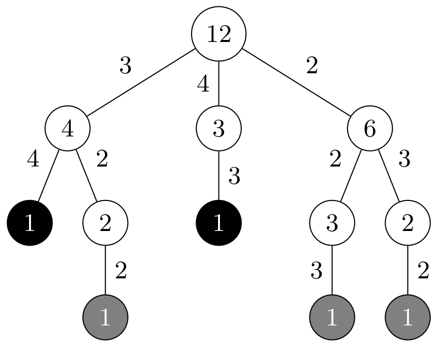
(a) Draw the game tree for $n = 12$. What is the best strategy for the first player?

(b) Play the number game using different values of n. Can you find a game (*i.e.*, an n) for which the second player always wins?

Solution:

(a) The tree looks as follows:

Gray nodes indicates a win of the first player, black nodes a win of the second player.



The best strategy for the first player is to divide by 2.

(b) If $n = pq$ for two primes p, q the second player always wins.