Algorithmic Discrete Mathematics
6. Exercise Sheet

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Groupwork

Exercise G1
We consider an application of Hall's Marriage Theorem:

Let $X = \{S_i \mid 1 \leq i \leq k\}$ be a finite family of sets. A *system of distinct representatives (SDR)* or *transversal* of X is a set T with a bijection $\varphi : T \to X$ such that $t \in \varphi(t)$.

Prove the following theorem:

X has an SDR if and only if for any $j \in \{1, \ldots, k\}$ the union of any j of the S_i has size at least j.

Solution: Let $S = \bigcup S_j = \{a_1, \ldots, a_n\}$ for some n. We define a bipartite graph G on the node set $\{S_1, \ldots, S_k\} \cup \{a_1, \ldots, a_n\}$ with an edge between S_i and a_j if and only if $a_j \in S_i$.

A transversal then corresponds to a matching in G of size k. By the Marriage Theorem there is a transversal if and only if $|A| \geq |N(A)|$ for all $A \subseteq X$. This is equivalent to the condition in the theorem.

Exercise G2
For each of the following families of sets determine whether the condition of the theorem on SDRs is met. If so, then find an SDR and the corresponding bijection φ. If not, then show how the condition is violated.

(a) $\{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5\}, \{1,2,5\}$

(b) $\{1,2,4\}, \{2,4\}, \{2,3\}, \{1,2,3\}$

(c) $\{1,2\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{1,3\}, \{3,4\}$

(d) $\{1,2,5\}, \{1,5\}, \{1,2\}, \{2,5\}$

(e) $\{1,2,3\}, \{1,2\}, \{1,3\}, \{1,2,3,4,5\}, \{2,3\}$

Solution: We list the representatives in the order of the sets. *I.e.,* we list $\varphi^{-1}(S_1), \varphi^{-1}(S_2), \ldots$

(a) SDR: $1,2,3,4,5$

(b) SDR: $1,4,2,3$

(c) Here $|X| = 6$ but $|\bigcup S_i| = |\{1,2,3,4\}| = 4$.

(d) Here $|X| = 4$ but $|\bigcup S_i| = |\{1,2,5\}| = 3$.

(e) Here $|S_1 \cup S_2 \cup S_3 \cup S_5| = |\{1,2,3\}| = 3$ but $|\{S_1, S_2, S_3, S_5\}| = 4$.

Exercise G3
Consider the following problem: Assume that there are n factories, producing a supply of $s_1, \ldots, s_n$ of some good. Moreover, there are m customers, each asking for a demand of $d_1, \ldots, d_m$. Each factory i can deliver an amount $c_{ij} \geq 0$ to the customer j.

(a) Model the problem as a flow problem.

(b) Now consider the following real world problem: There are six universities that will produce five mathematics graduates each. Moreover, there are five companies that will be hiring $7,7,6,6,5$ math graduates, respectively. No company will hire more than one student from any given university. Will everyone get a job?

Solution:

(a) We construct a directed graph $G = (V, E)$ with

$$V = \{s, t\} \cup \{f_1, \dots, f_n\} \cup \{c_1, \dots, c_m\}$$

(*i.e.*, one node for each factory and customer and additional source and sink nodes) and

$$E = \{(s, f_i) \mid i = 1, \dots, n\} \cup \{(c_j, t) \mid j = 1, \dots, m\} \cup \{(f_i, c_j) \mid i \in [n], j \in [m]\}$$

and capacity $c : E \to \mathbb{R}$:

$$\begin{aligned} c(s, f_i) &= s_i, & i &= 1, \dots, n \\ c(c_j, t) &= d_j, & j &= 1, \dots, m \\ c(f_i, c_j) &= c_{ij}, & i &\in [n], j \in [m]. \end{aligned}$$

The demands can be satisfied if and only if there is a flow in G of value $\sum d_j$.

(b) Again we can model this as a flow problem. Let $u_1, \dots, u_6$ be the nodes corresponding to the universities and $c_1, \dots, c_5$ those corresponding to the companies. The cut $\{s, u_1, \dots u_6, c_5\}$ has capacity 29. Hence not everyone will get a job.

Exercise G4

We want to construct an $(n \times m)$-matrix whose entries are nonnegative integers such that the sum of the entries in row i is r_i and the sum of the entries in column j is c_j. (Then clearly $\sum r_i = \sum c_j$.)

(a) What other constraints (if any) should be imposed on the r_i and c_j to assure such a matrix exists?

(b) Construct such a (5×6)-matrix with row sums $20, 40, 10, 13, 25$ and column sums all equal to 18.

Solution:

(a) We do not need any other constraints. We can construct this matrix as follows: Construct a network with graph $G = (V, E)$ with $V = \{R_1, \dots, R_n\} \uplus \{C_1, \dots, C_m\} \uplus \{s, t\}$ and

$$E = \{(s, R_i) \mid i \in [n]\} \cup \{(R_i, C_j) \mid i \in [n], j \in [m]\} \cup \{(C_j, t) \mid j \in [m]\}$$

and capacity

$$\begin{aligned} c(s, R_i) &= r_i, \\ c(R_i, C_j) &= \infty, \\ c(C_j, t) &= c_j. \end{aligned}$$

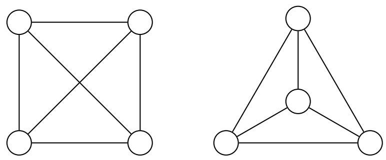
Then such a matrix corresponds to a maximal flow $f : E \to \mathbb{R}$ in this network. The entry in position (i, j) equals $f(R_i, C_j)$.

(b) Easy.

Exercise G5

A graph is *planar* if it can be drawn (or *embedded*) in the plane without intersecting edges.

For example, consider the graph K_4. The first drawing is not planar, the second one is:



So K_4 is a planar graph.

An embedding of a planar graph subdivides $\mathbb{R}^2$ into connected components, the *faces*. *E.g.* the planar drawing of K_4 above has four faces, three bounded triangular ones and one unbounded face.

Prove *Euler's Formula* for a connected planar graph $G = (V, E)$ with $|V|$ vertices, $|E|$ edges and $F(G)$ faces:

$$|V| - |E| + F(G) = 2.$$

Solution: We prove this by induction on $|E|$. If $|E|=1$ then $|V|=2$ and $F(G)=1$ since there is only one graph with one edge.

Now assume the claim is true for all graphs with $|E|\leq n$. Then there are two cases:

- Either there is a cycle in G. Then let e be any edge contained in this cycle. Let G' be the graph obtained from G by removing e. Then G' is connected but the number of faces is reduced by one. *I.e.*, $F(G')=F(G)-1$ and thus
$$|V|-|E|+F(G)=|V(G')|-(|E(G')|+1)+(F(G')+1)=|V(G')|-|E(G')|+F(G')=2.$$
- If there is no cycle in G, then G is a tree and hence has a leaf v. Let e be the edge incident to v. Let G' be the graph obtained from G by removing e and v. Then $F(G)=F(G')$ and hence
$$|V|-|E|+F(G)=(|V(G')|+1)-(|E(G')|+1)+F(G')=|V(G')|-|E(G')|+F(G')=2.$$

Homework

Exercise H1 (5 points)
Determine the number of perfect matchings in

(a) $K_{n,n}$ and

(b) K_{2n}.

Solution:

(a) $n!$

(b) $\prod_{i=1}^{n}(2n-2i+1)$

Exercise H2 (5 points)
Let $G=(A\cup B,E)$ be a d-regular bipartite graph for $1\leq d$, *i.e.,* all vertices have degree d.

(a) Show that $|A|=|B|$.

(b) Show that G contains a perfect matching.

(c) Using induction, prove that the edges of G can be partitioned into d perfect matchings.

Solution:

(a) Since each edge connects the two sets A and B, the sums of the degrees have to agree, *i.e.,*
$$d|A|=\sum_{v\in A}\deg v=\sum_{v\in B}\deg v=d|B|$$
and hence $|A|=|B|$.

(b) We want to apply Hall's Theorem and therefore have to prove that G satisfies the condition
$$|N(S)|\geq |S|$$
for each $S\subseteq A$. To this end, let S be a k-subset of A. There are dk edges $\{u,w\}$ with $u\in S, w\in B$. As any vertex of B is incident with exactly d edges, those dk edges have to be incident with at least k distinct vertices in B, which implies the condition. Therefore, by Hall's Theorem, G has a perfect matching.

(c) We will use induction over d. For $d=1$ this is true by (b). Now assume $d\geq 2$. Again by (b) there is a perfect matching. If we remove this, we obtain a $(d-1)$-regular graph, which by induction can be decomposed into $d-1$ perfect matchings.

Exercise H3 (5 points)
A *(minimal) node cover* of a graph $G=(V,E)$ is a subset $U\subseteq V$ (with minimal cardinality) such that for every $\{v,w\}\in E$ at least one of v,w is contained in U, *i.e.,* $\{v,w\}\cap U\neq\emptyset$).

(a) Prove the *Theorem of König*:
In a bipartite graph the size ν of a maximal matching equals the size τ of a minimal node cover.

(b) Conclude that a bipartite graph has a perfect matching if and only if every node cover has size at least $\frac{1}{2}|V|$.

Solution:

(a) Let $G = (A \dot\cup B, E)$ be an undirected bipartite graph. It is clear that any node cover has at least the cardinality of a maximal matching, hence $\tau \geq \nu$.
Conversely, by Proposition 6.11 there is $S \subseteq A$ such that

$$\nu = |A| - |S| + |N(S)| = |A \setminus S| + |N(S)|.$$

But $A \setminus S \cup N(S)$ is a node cover, so $\tau \leq \nu$.

(b) First assume that $G = (A \dot\cup B, E)$ has a perfect matching. Then $\nu = |A| = |B| = \frac{1}{2}|V|$. By König's Theorem $\tau = \nu = \frac{1}{2}$.
Conversely assume that $\tau \geq \frac{1}{2}|V|$. Then also $\nu \geq \frac{1}{2}|V|$. But the size of a matching in a bipartite graph is bounded from above by $\min(|A|, |B|) \leq \frac{1}{2}|V|$.

Exercise H4 (5 points)

(a) Show that every graph with a least six vertices contains the graph K_3 or its complement $\overline{K_3}$.

(b) Show that every graph with a least ten vertices contains K_4 or $\overline{K_3}$.

(c) Show that the assertion in (b) does not hold for graphs with eight vertices.

Solution:

(a) Let $v_1, \ldots, v_6$ be the vertices of the graph G. Then (by the pigeon hole principle) v_1 is connected to at least three other vertices in either G and $\overline{G}$. So assume without loss of generality that v_1 is connected to v_2, v_3, v_4 in G. If one of the three edges $\{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}$ is contained in G, then G contains a K_3. Otherwise v_2, v_3, v_4 form a $\overline{K_3}$.

(b) Let $v_1, \ldots, v_{10}$ be the vertices of G. Consider the two cases:
- There is vertex v that has degree at least 6 in either G or $\overline{G}$. Then by (a) among the neighbors of v there is either a K_3, which together with v yields a K_4, or a $\overline{K_3}$.
- The maximal degree in G and $\overline{G}$ equals 5. W.l.o.g. let $\deg v_1 = 5$ and $N(v_1) = \{v_2, \ldots, v_6\}$. Then either there is an edge between any two of $v_7, \ldots, v_{10}$ (yielding a K_4) or (if there is no edge between v_j and v_k, $j, k \in \{7, \ldots, 10\}$) then v_1, v_j, v_k induce a $\overline{K_3}$.

(c)