Algorithmic Discrete Mathematics
1. Exercise Sheet

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Groupwork

Exercise G1
Show that a graph $G = (V, E)$ is bipartite if and only if it does not contain cycles of odd length.

Solution:

"⇒" If G is bipartite with color classes A and B, then any path in G alternates between A and B. Hence every cycle has even length.

"⇐" Let G be a graph that does not contain an odd cycle. We may w.l.o.g. assume that G is connected. Now fix some vertex $v \in V$. Define $A \subset V$ to be the set of all vertices w such that there is a path of odd length in G from v to w. Moreover, we define $B = V \setminus A$. (Then B contains all vertices with even distance from v.) It now suffices to show that $\binom{A}{2} \cap E, \binom{B}{2} \cap E = \emptyset$, *i.e.,* there are no edges between any vertices of A, respectively B.

Assume that there is an edge $\{a, a'\}$ with $a, a' \in A$. Then there are a $[v, a]$-path p_1 and a $[v, a']$-path p_2 of odd length. We can then construct a closed walk $C = (v, p_1 a, a', p_2, v)$ of odd length, which contains some odd cycle. This contradicts our assumption.

The proof that there is no edge $\{b, b'\}$ with $b, b' \in B$ is similar.

Exercise G2
Let $G = (V, E)$ be a graph. Prove:

(a) Any walk with distinct endpoints v, w contains a path between v and w.

(b) Any closed walk contains a cycle.

Solution:

(a) Let p be a walk between v and w. If p is a path, we are done. Otherwise there is some vertex x that occurs twice along p. We delete everything between the two occurrences and one of the x to get a new walk p'. We then repeat this procedure until we obtain a path.

(b) The cycle is constructed similarly to the path in (a) except that we ignore the endpoint.

Exercise G3
A walk in a connected graph $G = (V, E)$ is called an *Eulerian trail* if it contains each edge of G exactly once. A closed Eulerian trail is called an *Eulerian tour*. The graph G is called *Eulerian* if it contains an *Eulerian tour*.

(a) Which of the graphs in Figure 1 are Eulerian.

(b) Let G be a connected graph. State conditions for G to be Eulerian and prove that these conditions are necessary.

(c) Are these conditions also sufficient?

Solution:

(a) no, yes, yes, no

(b) A necessary condition is: Every vertex has even degree. This is necessary, since along the cycle whenever we arrive at a vertex, we have to leave it again. *I.e.,* the edges incident to a vertex come in pairs.

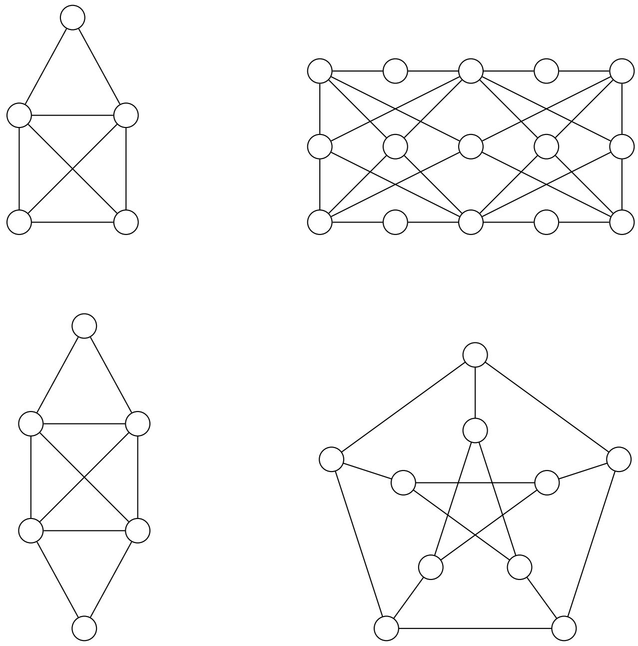

Figure 1: Eulerian or not?

(c) We show that the condition from (b) is sufficient by constructing an Eulerian tour.

We first pick some vertex v. Then we begin some walk in v along "unused" edges of G until we cannot go any further. Since every vertex has even degree, the constructed walk p ends in v. If this is an Eulerian tour, we are done. Otherwise there is some vertex v' in p that has unused edges left. We then start a new walk p' in v'. Again, this has to end in v'. We can then extend p to a walk p_1 of the form $p_1 = (v, \dots, v', p', v', \dots, v)$. If we repeat this process until no unused edges are left, we obtain an Eulerian tour.

Homework

Exercise H1 (5 points)
Show: In any graph with at least two vertices, there are at least two vertices with the same degree.

Solution: Let G be a graph with n vertices. The minimal degree of a vertex is 0, the maximal degree is $n-1$. Since not both 0 and $n-1$ can occur as degrees, there are at most $n-1$ different degrees. Thus, (by the pigeon hole principle) at least two vertices have the same degree.

Exercise H2 (5 points)
For a graph $G = (V, E)$, the graph $\overline{G} = (V, \binom{V}{2} \setminus E)$ is called the *complementary graph* of G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.
Show: One of G and $\overline{G}$ is connected.

Solution: Assume that G is not connected. (Otherwise we are done.) Let $V' \subset V$ be the vertex set of some connected component of G. Fix $v \in V \setminus V'$. We have to show that there is a path from v to any other vertex $w \in V$. If $w \in V'$, then $\overline{G}$ contains the edge $\{v, w\}$. Otherwise, choose some vertex $v' \in V'$. Then $\overline{G}$ contains the edges $\{v, v'\}$ and $\{v', w\}$. Hence (v, v', w) is a $[v, w]$-path in $\overline{G}$. This shows that $\overline{G}$ is connected.

Exercise H3 (10 points)
Show that for a graph $G = (V, E)$ with $n \geq 2$ vertices the following are equivalent:

(i) G is a tree.

(ii) G is connected and contains $n-1$ edges.

(iii) G contains $n-1$ edges, but no cycle.

(iv) G is *minimally connected*, *i.e.*, G is connected but $G - e$ is not connected for any $e \in E$.

(v) G is *maximally acyclic*, *i.e.*, G is acyclic but $G + e$ contains a cycle for any $e \in \binom{V}{2} \setminus E$.

(vi) For each pair $u, v \in V$ of vertices, there is a unique $[u, v]$-path in G.

Solution: We show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (i)$:

$(i) \Rightarrow (ii)$: We show that G has $n-1$ edges by induction over n. If $n=1$ then G has one vertex and 0 edges. Now let G be a tree with $n>1$ vertices and let v be a leave of G. We remove v and its incident edge to obtain a tree G' with $n-1$ vertices. By induction G' has $n-2$ edges and hence G has $n-1$ edges.

$(ii) \Rightarrow (iii)$: Again, we show by induction over n that G has no cycle. If $n=1$ then G has no edges and only one vertex. This graph is obviously acyclic. Now assume that G has $n>1$ vertices and $n-1$ edges and is connected. By connectedness every vertex has degree at least one. If $\deg v \geq 2$ for every vertex v then $|E| = \frac{1}{2}\sum \deg v \geq n$. Hence G has a leave v. If we remove v and its incident edge, we obtain a connected graph G' with $n-1$ vertices and $n-2$ edges. By induction G' does not contain any cycle. Thus, G is also acyclic.

$(iii) \Rightarrow (iv)$: We first show (by induction over n) that G is connected. If $n=1$, then G is obviously connected. Now let $n>1$. Since G has no cycle, there is a longest path in G, whose endpoints are leaves. By removing a leave v and its incident edge, we obtain a connected graph G' with $n-1$ vertices and $n-2$ edges. By induction G' is connected and by construction G is so, too.

If there is no cycle, then there is at most one path between any two vertices. So removing any edge will make the graph disconnected. Hence G is minimally connected.

$(iv) \Rightarrow (v)$: We first show that G is acyclic. Suppose on the contrary that there is a cycle. Then we can remove any edge from this cycle without affecting connectedness. Hence G is acyclic.

Now fix an edge $e = uv \notin E$. Since G is connected there is a path p from u to v in G. Hence $G+e$ contains a cycle and thus G is maximally acyclic.

$(v) \Rightarrow (vi)$: Let $u, v \in V$. Since G is maximally acyclic there is a $[u,v]$-path. Otherwise we could add the edge uv without creating a cycle. On the other hand if there were two $[u,v]$-paths p_1, p_2 then the concatenation of p_1 and the reverse of p_2 forms a closed walk with startpoint u. By G2 this contains a cycle. Hence there is a unique $[u,v]$-path.

$(vi) \Rightarrow (i)$: G is connected since for any $u, v \in V$ there is a $[u,v]$-path in G. Moreover, G is acyclic since this path is unique. (The line argument is as in the previous part.)