Topological Groups
6. Exercise Sheet

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Groupwork

Exercise G1 (Integrals)
Let G be compact abelian, $\chi \in \hat{G}$ a non-trivial character and let μ be a/the Haar-measure of G. Show that:

$$\int_{\mathbb{G}} \chi(g) d\mu(g) = 0.$$

What happens with the value of $\int_G \chi(g) d\mu(g)$ if χ is trivial?

Exercise G2 (Divisibility)
Let $\frac{1}{p^\infty}\mathbb{Z}$ be the (additive) subgroup of $\mathbb{Q}$ consisting of elements which be may be written in the form $\frac{m}{p^n}$, where $m \in \mathbb{Z}, n \in \mathbb{N}$. Define $\mathbb{Z}(p^\infty) := \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z}$. Show: $\mathbb{Z}(p^\infty) := \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z}$ is divisible.

Hint: Show first that every element in $\mathbb{Z}(p^\infty)$ is of order p^m for some m. Conclude that the identity $p^k x = g$ has a solution in $\mathbb{Z}(p^\infty)$ for all $g \in G$. Finally, show that in a finite abelian group of order m the identity $nx = g$ has a solution for every n which is coprime to m. Now you're done, why?

Definition: For p a prime the group $\mathbb{Z}(p^\infty)$ is called **Prüfer-group**.

Exercise G3 (More hats)
Let $\varphi : A \to B$ be a morphism. This induces a morphism $\hat{\varphi} : \hat{B} \to \hat{A}$ via $\hat{\varphi}(\chi) = \chi \circ \varphi$.

Show: For every morphism $f : A \to \hat{G}$ there exists a morphism $f' : G \to \hat{A}$ such that $f = \hat{f}' \circ \eta_A$.

Hint: Define $f'(g)(a) := f(a)(g)$ and apply the definitions.

Conversely, for every morphism $f : G \to \hat{A}$ there exists a morphism $f' : A \to \hat{G}$ such that $f = \hat{f}' \circ \eta_G$

Exercise G4 (Even more hats)
Let A be abelian, then $\hat{\eta}_A \circ \eta_{\hat{A}} = \mathrm{id}_{\hat{A}}$ and for a compact abelian group G we have $\hat{\eta}_G \circ \eta_{\hat{G}} = \mathrm{id}_{\hat{G}}$

Exercise G5 (Recently in the hat shop...)
If A is abelian and η_A is an isomorphism, then $\eta_{\hat{A}}$ is an isomorphism as well. Similarly, if G is compact abelian and η_G an isomorphism, then $\eta_{\hat{G}}$ is an isomorphism.