Topological Groups
3. Exercise Sheet

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Groupwork

Exercise G1 (Open Mapping Theorems)
Let τ_{disc} be the discrete and τ the usual order topology on $\mathbb{R}$. The map $\mathrm{id}\colon (\mathbb{R}, \tau_{disc}) \to (\mathbb{R}, \tau)$ is continuous, but obviously fails to be open. Why are the Open Mapping Theorems not applicable?

Exercise G2 (Morphisms)
Show that a homomorphism $f : G \to H$ is continuous and open if and only if $f(\mathfrak{U}_G) = \mathfrak{U}_H$.

Exercise G3 (Totally Disconnected Groups)
Let G be a connected group and let N be a totally disconnected normal subgroup of G. Show that $N \leq Z(G)$, i.e. every element of N is central in G.

Exercise G4 (Connected Groups)
Let G be a connected group and let $U \in \mathfrak{U}$ be an identity neighbourhood. Prove or give a counterexample: $G = \langle U \rangle$.

Homework

Exercise H1 (Locally Compact Groups)
Let G be locally compact and connected. Then there exists a compact identity neighbourhood $K \subseteq G$ such that $G = \langle K \rangle$. In particular, connected locally compact groups are compactly generated.

Exercise H2 (Open Subgroups)
Let G be a topological group. Show that a subgroup is open if and only if it is clopen if and only if it contains an interior point.

Exercise H3 (Vector Spaces)
Let V be a real vector space with two norms $||.||_1$ and $||.||_2$ such that V becomes complete with respect to both norms. Assume that there exists a constant $c > 0$ such that $||x||_1 \leq c||x||_2$ for all $x \in V$. Show that the two topologies induced by the norm coincide.