Topological Groups
2. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Andreas Mars

Summer Term 2013
8.5.2013

Groupwork

Exercise G1 (Identity Neighbourhoods)
Let G be a topological group. Show:

(a) For any identity-neighbourhood $U \in \mathfrak{U}$ there exists an (open) identity-neighbourhood V with the properties:
$$VV \subseteq U, \quad V^{-1}V \subseteq U, \quad VV^{-1} \subseteq U, \quad V^{-1}V^{-1} \subseteq U.$$
(b) The multiplication map $\mu_G \colon G \times G \to G$ is open.
(c) Let $Y \subseteq G$ be a connected subset. Then $\overline{Y}$ is connected as well. (NB: This is true for all topological spaces.)

Exercise G2 (Normal Subgroups)
Let G be an abstract group and let $N \lhd G$ be a normal subgroup. Assume that N is a topological group and the action of G on N by conjugation is continuous. Show that there exists a unique group topology on G such that

(a) the induced topology on N coincides with the original topology and
(b) N is open in G.

You might want to prove the following Lemma along the way: A subgroup $H \leq G$ is open if and only if it contains an identity neighbourhood.

Exercise G3 (Closed Sets)
Let $f : G \to H$ be a morphism and let H be Hausdorff. Then the graph $X := \{(x, y) \in G \times H \mid f(x) = y\}$ of f is closed in $G \times H$.

Hence or otherwise, show that for any family of Hausdorff groups X_n and $f_n \colon X_{n+1} \to X_n$ morphisms the set $\{(x_n) \mid (\forall n \in \mathbb{N}) \colon f_n(x_{n+1}) = x_n\}$ is a closed subset of $\prod X_n$.

Exercise G4 (The Neighbourhood filter of the identity)

(a) Find an example of a topological group G and two closed subsets $A, B \subseteq G$ such that $AB \subseteq G$ is **not** closed (cf. Proposition 2.12 (ii)).
(b) Prove Corollary 2.16: Let G be a topological group and H be Hausdorff. Then for each morphism $f : G \to H$ there is a unique morphism $f' \colon G/\overline{\{1\}} \to H$ such that $f = f' \circ q$, where $q \colon G \to G/\overline{\{1\}}$ is the quotient morphism.

Homework

Exercise H1 (Decomposition of Morphisms)
Prove Proposition 2.10 (canonical decomposition of morphisms).

Exercise H2 (A Clopen Intersection)
Prove Lemma 2.19.

Exercise H3 (Characteristic Subgroups)
Prove or give a counterexample:

(a) Let $N \lhd G$ be a normal subgroup of G. Then N is characteristic.
(b) If G is abelian (in particular, every subgroup is normal), then every subgroup is characteristic.

Exercise H4 (Construction of Topologies)
In case Theorem 2.22 does not seem familiar to you, prove it.