Topological Groups
1. Exercise Sheet

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Groupwork

Exercise G1 (Homogeneous Spaces)
Give a proof or a counterexample:

(a) A discrete or indiscrete topological space is homogeneous.

(b) The open interval $(0,\infty) \subseteq \mathbb{R}$ with the order topology is homogeneous.

(c) The compact unit interval $[0,1] \subseteq \mathbb{R}$ with the order topology is homogeneous.

Exercise G2 (Topological Groups)

(a) Prove: The matrix groups $\mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ equipped with the subspace topology induced from $\mathbb{R}^{n\times n}$ resp. $\mathbb{C}^{n\times n}$ are topological groups.

(b) Show that all groups from part (a) are closed subgroups of $\mathrm{GL}_n(\mathbb{C})$. Is one of them an open subgroup?

(c) Show that for $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ the group $\mathrm{GL}_n(\mathbb{F})$ is open in $\mathbb{F}^{n\times n}$, while $\mathrm{SL}_n(\mathbb{F})$ is closed in $\mathbb{F}^{n\times n}$.

(d) Prove: A group G equipped with a topology τ is a topological group if and only if the map

$$G \times G \to G$$
$$(g,h) \mapsto gh^{-1}$$

is continuous with respect to τ.

(e) Let $H \leq G$ be an open subgroup. Give a proof or counterexample: Then H is a closed subgroup.

Exercise G3 (Subgroups and Quotients)
Show:

(a) Let $H \leq G$ be a subgroup of a topological group G. Then the map

$$(g, g'H) \mapsto gg'H$$

in continuous in the second argument.

(b) If G/H is Hausdorff, then H is a closed subgroup. Give necessary and sufficient condition(s) on H such that the quotient G/H is a topological group again.

Exercise G4 (Topological Manifolds)
Prove Proposition 1.6: A connected Hausdorff topological manifold X is homogeneous.

(a) Show that for every two points x, y in the open unit ball $B_1(0)$ of $\mathbb{R}^n$ there exists a homeomorphism of the closed unit ball $\overline{B_1(0)}$ which maps x to y and which fixes the boundary $\partial\overline{B_1(0)}$ pointwise.

(b) For $x \in X$ and U an open neighbourhood of x, there exists an open neighbourhood V of x with the property that $\overline{V} \subseteq U$ and the property that for all $v \in V$ there is a homeomorphism $f : V \to V$ with $f(x) = v$, which extends to a homeomorphism of X. Hint: Where can you use (i)?

(c) For arbitrary $x, y \in X$ there exists a compact line segment connecting x and y. Hint: Now you're done, using (ii). Why?

Homework

Exercise H1 (More Topological Groups)
Prove:

(a) The torus $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ equipped with usual complex multiplication and the subspace topology from $\mathbb{C}$ is a topological group.

(b) The space $\mathbb{R}/\mathbb{Z}$ with the quotient topology and the operation $(x, y) \mapsto x + y (\text{mod } 1)$ is a topological group.

(c) The exponential map $\exp\colon \mathbb{R}/\mathbb{Z} \to \mathbb{T}, x \mapsto e^{2\pi i x}$ is a group homomorphism which is bijective, continuous and open (i.e. an isomorphism of topological groups).

Exercise H2 (Construction of Topological Groups)
Prove Proposition 2.6:

(a) Let G be a topological group and let $H \leq G$ be a subgroup equipped with the subspace topology. Then H is a topological group.

(b) Let $\{G_i\}_{i \in I}$ be an arbitrary family of topological groups and equip the cartesian product $\prod_{i \in I} G_i$ with the usual product topology. Then $\prod_{i \in I} G_i$ is a topological group.

(c) Let $N \lhd G$ be a normal subgroup and equip G/N with the quotient topology. Then G/N is a topological group. Also try to show that G/N is Hausdorff if and only if $N = \overline{N}$.

Exercise H3 (A Quotient)
Consider the additive group $\mathbb{R}$ with the usual order topology and its subgroup $\mathbb{Q}$. Since $\mathbb{R}$ is abelian, every subgroup is normal, hence in particular $\mathbb{Q}$ is and we may consider the quotient group $G := \mathbb{R}/\mathbb{Q}$.

(a) Remind yourself that G is a topological group. Is it Hausdorff, T_1 and/or T_0?

(b) Prove or disprove: The subgroup $\{1\}$ is dense in G.

Hint: Compare the groups $G/\overline{\{1\}}$ and $\mathbb{R}/\overline{\mathbb{Q}}$. Do you notice something?