classical undecidability results for FO

$\mathrm{Th}(\mathfrak{N}) := \{\varphi \in \mathrm{FO}_0(\sigma_{\mathrm{ar}}) \colon \mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <) \models \varphi\}$

theorem (Tarski)

$\mathrm{Th}(\mathfrak{N})$ undecidable and not recursively axiomatisable

method: reduction from H

based on FO-definable arithmetical encoding of finite sequences over $\mathbb{N}$

Gödel's β for quantification over finite sequences

Gödels Incompleteness Theorems

Gödel's incompleteness theorems show that Hilbert's programme cannot be fulfilled, in a very strong sense

- reasonable FO-axiomatisations of sufficiently rich theories are necessarily incomplete and cannot prove their own consistency
- these limitations are 'limitations in principle'

method: self-reference & diagonalisation (Epimenides' liar)
via internalisation of notions of recursion and provability
in FO theories that support enough arithmetic

completeness & recursive axiomatisation

basic definitions:

a FO-theory $T \subseteq \mathrm{FO}_0(\sigma)$ is complete
if for all $\varphi \in \mathrm{FO}_0(\sigma)$, $\varphi \in T$ or $\neg\varphi \in T$

a FO-axiomatisation $\Phi \subseteq \mathrm{FO}_0(\sigma)$ is complete
if for all $\varphi \in \mathrm{FO}_0(\sigma)$, $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$

$T \subseteq \mathrm{FO}_0(\sigma)$ recursively axiomatisable
if $T = \Phi^{\vdash}$ for some recursive $\Phi \subseteq \mathrm{FO}_0(\sigma)$

remarks:

T complete and recursively axiomatisable $\Rightarrow$ T recursive

T has a recursive axiom system if, and only if,
T has a recursively enumerable axiom system

representativity

fix σ and $\Phi \subseteq \mathrm{FO}_0(\sigma)$ together with a recursive map for the representation of natural numbers by variable-free terms:

$$\left.\begin{array}{ccc} \mathbb{N} & \longrightarrow & T_\sigma(\emptyset) \\ n & \longmapsto & \underline{n} \end{array}\right\} \text{ such that } \Phi \vdash \neg\underline{n} = \underline{m} \text{ for all } n \neq m \in \mathbb{N}$$

- $\varphi(\mathbf{x})$ represents $R \subseteq \mathbb{N}^n$ if, f.a. $\mathbf{m} \in \mathbb{N}^n$,

  $$\begin{array}{lcl} \mathbf{m} \in R & \Rightarrow & \Phi \vdash \varphi(\underline{\mathbf{m}}) \\ \mathbf{m} \notin R & \Rightarrow & \Phi \vdash \neg\varphi(\underline{\mathbf{m}}) \end{array}$$

- $\varphi(\mathbf{x}, z)$ represents $f : \mathbb{N}^n \to \mathbb{N}$ if, f.a. $\mathbf{m} \in \mathbb{N}^n$,

  $$\Phi \vdash \exists^{=1} z\, \varphi(\underline{\mathbf{m}}, z) \wedge \varphi(\underline{\mathbf{m}}, \underline{f(\mathbf{m})})$$

examples of theories and representations

definition:
Φ admits representations if every total recursive function $f\colon \mathbb{N}^n \to \mathbb{N}$ (and every recursive $R \subseteq \mathbb{N}^n$) can be represented

examples:

- $\mathrm{Th}(\mathfrak{N})$, first-order Peano arithmetic, and Julia Robinson's finite $Q \subseteq \mathrm{Th}(\mathfrak{N})$, all with $n \mapsto \underline{n} = \underbrace{1 + \cdots + 1}_{n}$
- ZFC with $\underline{0} = \emptyset$, $\underline{n+1} = \underline{n} \cup \{\underline{n}\}$

Julia Robinson's weak arithmetical theory Q

$Q \subseteq \mathrm{Th}(\mathfrak{N})$:

$$\left.\begin{array}{l} \forall x\, x+1 \neq 0 \\ \forall x \forall y (x \neq y \to x+1 \neq y+1) \\ \forall x (x \neq 0 \to \exists y\, x = y+1) \end{array}\right\} \quad S$$

$$\left.\begin{array}{l} \forall x\, x+0 = x \\ \forall x \forall y (x + (y+1) = (x+y)+1) \end{array}\right\} \quad +$$

$$\left.\begin{array}{l} \forall x\, x \cdot 0 = 0 \\ \forall x \forall y (x \cdot (y+1) = (x \cdot y) + x) \end{array}\right\} \quad \cdot$$

self-reference: the fixpoint theorem

fix bijective, recursive Gödelisation $\ulcorner\ \urcorner : \mathrm{FO}(\sigma) \longrightarrow \mathbb{N}$, $\varphi \longmapsto \ulcorner\varphi\urcorner$

with recursive inverse $n \mapsto \varphi_n$

fixpoint thm

for $\Phi \subseteq \mathrm{FO}(\sigma)$ with representation and Gödelisation as above, find (recursively) for every $\psi(x) \in \mathrm{FO}(\sigma)$ a sentence $\varphi \in \mathrm{FO}_0(\sigma)$ with $\Phi \vdash \varphi \leftrightarrow \psi(\underline{\ulcorner\varphi\urcorner})$

Gödel's first incompleteness theorem

from fixpoint theorem obtain

thm:

if Φ admits representations and is consistent, then Φ cannot represent $T := \Phi^{\vdash}$; it follows that T is undecidable

Tarski's thm

for $\Phi = \mathrm{Th}(\mathfrak{N})$: $\mathrm{Th}(\mathfrak{N})$ not representable in $\mathrm{Th}(\mathfrak{N})$, "there is no arithmetical truth-predicate for arithmetic"

Gödel's first incompleteness theorem

if Φ admits representations, is consistent and recursive, then $T := \Phi^{\vdash}$ is incomplete