examples from logic (for decidable σ, w.l.o.g. σ_∞)

some decidable sets (languages):

T_σ, $\mathrm{FO}(\sigma)$, ...

ZFC (the set of axioms)

some computable functions:

$\mathrm{qr}(\varphi)$, $\mathrm{rk}(\varphi)$, $\mathrm{free}(\varphi)$, ...

substitution $(\varphi, \mathbf{x}, \mathbf{t}) \mapsto \varphi\frac{\mathbf{t}}{\mathbf{x}}$

truth values of $\varphi \in \mathrm{FO}_0(\sigma)$ over finite σ-structures

some r.e. sets (languages or relations):

$\mathrm{VAL}(\mathrm{FO}(\sigma)) = \{\varphi \in \mathrm{FO}(\sigma) \colon \models \varphi\}$

$\{(\varphi, \psi) \in \mathrm{FO}(\sigma)^2 \colon \varphi \models \psi\}$

$(\mathrm{ZFC})^\vdash = (\mathrm{ZFC})^\models$ (set theory: the set of consequences)

towards undecidability

Gödelisation of R-programs over fixed $\mathbb{A}$:

natural encoding of R-programs as $\mathbb{A}$-words

$$\begin{array}{rcl} \ulcorner\ \urcorner \colon \{\mathrm{P} \colon \mathrm{P} \text{ R-program over } \mathbb{A}\} & \longrightarrow & \mathbb{A}^* \\ \mathrm{P} & \longmapsto & \ulcorner \mathrm{P} \urcorner \end{array}$$

with the property that

- $\ulcorner\ \urcorner$ and its (partial) inverse are 'computable'
- any 'decidable'/'computable' relation/function on R-programs P is (R-)decidable/computable in terms of $\ulcorner \mathrm{P} \urcorner$
- so that there is a universal R-program U w.r.t. $\ulcorner\ \urcorner$ over $\mathbb{A}$:

$$\begin{array}{lcl} (\ulcorner \mathrm{P} \urcorner, \mathbf{u}) \xrightarrow{\mathrm{U}} \mathrm{STOP} & \text{and} & (\ulcorner \mathrm{P} \urcorner, \mathbf{u}) \xrightarrow{\mathrm{U}} \mathbf{w} \\ \text{iff} \quad \mathbf{u} \xrightarrow{\mathrm{P}} \mathrm{STOP} & & \text{iff} \quad \mathbf{u} \xrightarrow{\mathrm{P}} \mathbf{w} \end{array}$$

self-reference, halting problem

halting problem(s) for R-programs over $\mathbb{A}$:

$$H := \{ \ulcorner P \urcorner : \Box \xrightarrow{P} \text{STOP} \}$$

$$\widehat{H} := \{ \ulcorner P \urcorner : \ulcorner P \urcorner \xrightarrow{P} \text{STOP} \}$$

both undecidable (with a simple reduction between the two)

originally formulated for Turing machines (Alan Turing 1936)

classical undecidability results for FO

$$\text{FINSAT(FO)} = \{\varphi \in \text{FO}_0(\sigma_\infty) : \varphi \text{ satisfiable in a finite model } \}$$

$$\text{SAT(FO)} = \{\varphi \in \text{FO}_0(\sigma_\infty) : \varphi \text{ satisfiable } \}$$

clearly: $\text{FINSAT(FO)} \subsetneq \text{SAT(FO)}$
FINSAT(FO) r.e.; SAT(FO) co-r.e.

theorem (Trakhtenbrot)

FINSAT(FO) undecidable (not co-r.e.)

theorem (Church–Turing)

SAT(FO) undecidable (not r.e.)

method: reductions from $\text{H}/\bar{\text{H}}$

classical undecidability results for FO

$\mathrm{Th}(\mathfrak{N}) := \{\varphi \in \mathrm{FO}_0(\sigma_{\mathrm{ar}}) \colon \mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <) \models \varphi\}$

theorem (Tarski)

$\mathrm{Th}(\mathfrak{N})$ undecidable and not recursively axiomatisable

method: reduction from H

based on FO-definable arithmetical encoding of finite sequences over $\mathbb{N}$

Gödel's β & Chinese remainder thm