mathematics in a first-order framework ?

- ideal proof theory of FO comes at a price: completeness implies compactness
- capture standard mathematical practice, not in individual first-order structures, but in comprehensive set-theoretic framework

change of perspective:

from the local FO view for individual structures to global view of a set theoretic universe retaining FO for axiomatisation and reasoning

e.g., prove Dedekind's theorem in ZFC: the FO-theory ZFC implies that every 'internal model of Peano's axioms' is isomorphic to 'the internal realisation of the natural numbers on ω '

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39/44

ZFC – Zermelo–Fraenkel set theory with Choice

FO axiomatisation of a set-theoretic universe for mathematics

the \in -structure of *all* sets, axiomatised in FO($\{\in\}$)

just sets + extensionality + enough sets + foundation

the principle of extensionality:

(EXT) $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$

set existence postulates:

(SEP), (PAIR), (UNION), (POWER), (INFINITY), (REP), (AC)

a law&order axiom:

(FOUND): the set-theoretic universe is well-founded w.r.t. \in

ZFC: the axioms, overview

the principle of extensionality:

 $(\text{EXT}) \quad \forall x \forall y \big(\forall z \big(z \in x \leftrightarrow z \in y \big) \to x = y \big)$

set existence postulates of naive set theory:

(SEP): subsets $\{z \in x : ...\}$ (PAIR): pair sets $\{x, y\}$ (UNION): (set-)unions of sets (POWER): power sets

more specific set existence postulates:

(INFINITY): an inductive set, as a 'first infinite set'

(REP): images under definable operations

(AC): choice sets (the axiom of choice)

a law&order axiom:

(FOUND): the set-theoretic universe is well-founded w.r.t. \in

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examples: mathematics in ZFC

the natural numbers:

(INF) guarantees some inductive set, the intersection over all inductive sets exists and is inductive: *the* minimal inductive set ω

 $\emptyset \in \omega, \quad \omega \text{ is closed under } \left\{ \begin{array}{ccc} S \colon \omega & \longrightarrow & \omega \\ n & \longmapsto & n \cup \{n\} \end{array} \right.$

and $ZFC \models "(\omega, \emptyset, S)$ satisfies (P1),(P2),(P3)" $ZFC \models "Dedekind's$ Theorem" 41/44

examples: mathematics in ZFC

ordinals:

on(x) := "x is a transitive set that is well-ordered by \in "

 $\operatorname{ZFC} \models \operatorname{on}(\omega) \land \forall x (\operatorname{on}(x) \to \operatorname{on}(S(x)) \land \ldots$

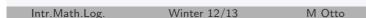
 ZFC implies that the ordinals form a proper class, are well-ordered by \in , represent all well-orderings of sets up to isomorphism, ...

 $\rm ZFC$ justifies definitions and proofs by ordinal recursion, \ldots

cardinals and cardinalities:

 $card(x) := "on(x) and x is not bijectively related to any <math>y \in x"$

ZFC implies that every set is bijectively related to a unique cardinal



mathematics in **ZFC**

empirical facts:

- standard mathematical practice can be modelled in ZFC
- ZFC provides a satisfactory FO framework

apparent puzzles:

- if ZFC is consistent, it must have a countable model
- if ZFC has a model, it must have non-standard models
- is ZFC part of mathematics or mathematics part of ZFC?

real questions:

• is ZFC consistent, and if so, can we show this?

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