mathematics in a first-order framework ?

- ideal proof theory of FO comes at a price: completeness implies compactness
- capture standard mathematical practice, not in individual first-order structures, but in comprehensive set-theoretic framework

change of perspective:
from the local FO view for individual structures
to global view of a set theoretic universe
retaining FO for axiomatisation and reasoning

e.g., prove Dedekind's theorem in ZFC:
the FO-theory ZFC implies that every 'internal model of Peano's axioms' is isomorphic to 'the internal realisation of the natural numbers on ω'

ZFC – Zermelo–Fraenkel set theory with Choice

FO axiomatisation of a set-theoretic universe for mathematics

the $\in$-structure of *all* sets, axiomatised in $\mathrm{FO}(\{\in\})$

just sets + extensionality + enough sets + foundation

the principle of extensionality:

$$(\mathrm{EXT}) \quad \forall x \forall y \big(\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y\big)$$

set existence postulates:

(SEP), (PAIR), (UNION), (POWER),
(INFINITY), (REP), (AC)

a law&order axiom:

(FOUND): the set-theoretic universe is well-founded w.r.t. $\in$

ZFC: the axioms, overview

the principle of extensionality:

(EXT) $\forall x \forall y \big(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y\big)$

set existence postulates of naive set theory:

(SEP): subsets $\{z \in x \colon \dots\}$
(PAIR): pair sets $\{x, y\}$
(UNION): (set-)unions of sets
(POWER): power sets

more specific set existence postulates:

(INFINITY): an inductive set, as a 'first infinite set'
(REP): images under definable operations
(AC): choice sets (the axiom of choice)

a law&order axiom:

(FOUND): the set-theoretic universe is well-founded w.r.t. $\in$

examples: mathematics in ZFC

the natural numbers:

(INF) guarantees some inductive set, the intersection over all inductive sets exists and is inductive: *the* minimal inductive set ω

$$\emptyset \in \omega, \quad \omega \text{ is closed under } \left\{ \begin{array}{rcl} S \colon \omega & \longrightarrow & \omega \\ n & \longmapsto & n \cup \{n\} \end{array} \right.$$

and $\mathrm{ZFC} \models$ "$(\omega, \emptyset, S)$ satisfies (P1),(P2),(P3)"

$\mathrm{ZFC} \models$ "Dedekind's Theorem"

examples: mathematics in ZFC

ordinals:

$\mathrm{on}(x) :=$ "x is a transitive set that is well-ordered by $\in$"

$\mathrm{ZFC} \models \mathrm{on}(\omega) \wedge \forall x(\mathrm{on}(x) \rightarrow \mathrm{on}(S(x)) \wedge \ldots$

ZFC implies that the ordinals form a proper class, are well-ordered by $\in$, represent all well-orderings of sets up to isomorphism, ...

ZFC justifies definitions and proofs by ordinal recursion, ...

cardinals and cardinalities:

$\mathrm{card}(x) :=$ "$\mathrm{on}(x)$ and x is not bijectively related to any $y \in x$"

ZFC implies that every set is bijectively related to a unique cardinal

mathematics in ZFC

empirical facts:

- standard mathematical practice can be modelled in ZFC
- ZFC provides a satisfactory FO framework

apparent puzzles:

– if ZFC is consistent, it must have a countable model
– if ZFC has a model, it must have non-standard models
– is ZFC part of mathematics or mathematics part of ZFC?

real questions:

- is ZFC consistent, and if so, can we show this?