consequences

- adequacy of a syntactic calculus (our sequent calculus) for all FO-based mathematical reasoning
- finite syntactic certificates (formal proofs) for all FO truths; recursive enumerability of all validities

example: FO group theory,

$$\{\varphi \in \mathrm{FO}_0(\{\circ, e\}) \colon \{\varphi_{\mathrm{G1}}, \varphi_{\mathrm{G2}}, \varphi_{\mathrm{G3}}\} \models \varphi\}$$
$$= \{\varphi_{\mathrm{G1}}, \varphi_{\mathrm{G2}}, \varphi_{\mathrm{G3}}\}^{\vdash} \subseteq \mathrm{FO}_0(\{\circ, e\})$$

can be algorithmically generated (r.e.)

model-theoretic consequences

- **compactness: finiteness property for satisfiability (!)**
- **Löwenheim–Skolem theorems:**

  (↓) countable consistent FO theories have countable models

  (↑) FO theories with infinite models have models of arbitrarily large cardinalities

and further, from these:

- *no* infinite structure $\mathfrak{A}$ is fixed up isomorphism by its FO theory $\mathrm{Th}(\mathfrak{A}) = \{\varphi \in \mathrm{FO}_0 \colon \mathfrak{A} \models \varphi\}$
- weaknesses/strengths of first-order logic/model theory: non-standard models, saturated models, ... richness of classical model theory, ...

compactness

$\Phi \subseteq \mathrm{FO}$ satisfiable if every finite subset $\Phi_0 \subseteq \Phi$ is satisfiable

- a finiteness property for satisfiability
- also a topological compactness assertion
- *the* tool (for model construction) in classical model theory

from finiteness property for consistency, via completeness

variants:

Φ unsatisfiable $\Rightarrow$ some finite $\Phi_0 \subseteq \Phi$ unsatisfiable

$\Phi \models \varphi \quad \Rightarrow \quad \Phi_0 \models \varphi$ for some finite $\Phi_0 \subseteq \Phi$

Löwenheim–Skolem theorems

for FO-theories $\Phi \subseteq \mathrm{FO}_0(\sigma)$:

(↓) Φ countable and satisfiable $\Rightarrow$
Φ has a countable model

(↑) Φ has an infinite model $\Rightarrow$
Φ has models in arbitrarily large cardinality

corollary: no FO-theory can determine any infinite structure up to isomorphism

non-standard models

$\mathrm{Th}(\mathfrak{A}) = \{\varphi \in \mathrm{FO}_0(\sigma) \colon \mathfrak{A} \models \varphi\}$

the *complete* FO-theory of σ-structure $\mathfrak{A}$

for familiar infinite standard structures $\mathfrak{A}$ of mathematics,

$\mathfrak{A}^* \models \mathrm{Th}(\mathfrak{A})$ with $\mathfrak{A}^* \not\simeq \mathfrak{A}$

is a non-standard companion of $\mathfrak{A}$:

indistinguishable from $\mathfrak{A}$ in FO,
but different – possibly in useful ways,
especially if $\mathfrak{A} \subseteq \mathfrak{A}^*$ and even $\mathfrak{A} \preccurlyeq \mathfrak{A}^*$

examples: non-standard models of natural and real arithmetic with 'infinite numbers' and 'infinitesimals'

example: non-standard analysis

find non-standard models of (expansions of)
real arithmetic $\mathfrak{R} = (\mathbb{R}, +, \cdot, 0, 1, <, \ldots)$

$\mathfrak{R}^* \succcurlyeq \mathfrak{R}$

with infinitesimals $\delta \in \bigcap_{1 \leqslant n \in \mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) \setminus \{0\}$

non-archimedean, Dedekind incomplete, real-closed field
with projection map to 'standard part' on $\bigcup_{n \in \mathbb{N}} (-n, n)$
allows to eliminate typical limit constructions of analysis

$\rightsquigarrow$ non-standard analysis, following Abraham Robinson (1960s)