Gödel's Completeness Theorem

the sequent calculus $\mathcal{S}$ is sound and complete for FO, i.e., for every $\mathrm{FO}(\sigma)$-sequent $\Gamma\ \varphi$ (in the restricted syntax with $=, \neg, \vee, \exists$):

$\Gamma\ \varphi$ derivable in $\mathcal{S}$: $\Gamma \vdash \varphi$ iff $\Gamma\ \varphi$ valid: $\Gamma \models \varphi$

strong form (for completeness claim proper):

$\Phi \models \varphi$ implies $\Phi \vdash \varphi$

for all $\Phi \subseteq \mathrm{FO}(\sigma), \varphi \in \mathrm{FO}(\sigma)$
including infinite sets Φ

towards the completeness proof

reduction: it suffices to show: **$\Phi \not\vdash \bot$ implies Φ satisfiable**
i.e., to provide models for consistent sets $\Phi \subseteq \mathrm{FO}$

Henkin construction: obtain a model from syntactic material;
a term model based on a quotient of an expansion of the term structure $\mathfrak{T}_\sigma$ to a suitable σ-interpretation

preparation: replace Φ by maximally consistent superset
with witness terms for existential assertions

analysis of consistency
inspection of the calculus, derived rules, ...

reduction to consistency/satisfiability

Φ inconsistent if, for some (indeed, every) φ:
$\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$
the syntactic counterpart of unsatisfiability

- $\Phi \models \varphi$ iff $\Phi \cup \{\neg\varphi\}$ unsatisfiable
- $\Phi \vdash \varphi$ iff $\Phi \cup \{\neg\varphi\}$ inconsistent

views of completeness:

syntactic	semantic
$\Phi \vdash \varphi$	$\Phi \models \varphi$
$\vdash \varphi$	$\models \varphi$
consistency	satisfiability
provability	validity

Henkin structures

a σ-interpretation for any $\Phi \subseteq \mathrm{FO}(\sigma)$:

$$\mathfrak{H} = \mathfrak{H}(\Phi) = \big(\mathbf{T}_\sigma/{\sim}, (\mathbf{f}^{\mathfrak{H}}), (\mathbf{R}^{\mathfrak{H}}), (\mathbf{c}^{\mathfrak{H}}), \beta^{\mathfrak{H}}\big)$$

- the relation $\quad t \sim t' \;:\Leftrightarrow\; \varphi \vdash t = t'$
  is a congruence w.r.t. $\mathfrak{T}_\sigma$ on T_σ (cf. equality rules of $\mathcal{S}$)
  $\rightsquigarrow$ $(\mathfrak{T}_\sigma/{\sim}, \beta) = \big(T_\sigma/{\sim}, (f^{\mathfrak{T}_\sigma}/{\sim}), (c^{\mathfrak{T}_\sigma}), (\beta \colon x \mapsto x/{\sim})\big)$
  a well-defined σ_{fctn}-interpretation

- $R^{\mathfrak{H}} := \big\{(t_1/{\sim}, \ldots, t_n/{\sim}) \colon \Phi \vdash Rt_1 \ldots t_n\big\}$
  $\rightsquigarrow$ $\mathfrak{H} := \big(T_\sigma/{\sim}, (f^{\mathfrak{T}_\sigma}/{\sim}), (R^{\mathfrak{H}}), (c^{\mathfrak{T}_\sigma}), \beta\big)$
  a well-defined σ-interpretation

- for any Φ, $\mathfrak{H}$ satisfies $\quad \mathfrak{H} \models \alpha \;\Leftrightarrow\; \Phi \vdash \alpha \;$ for all atomic α

Henkin models

for any set $\Phi \subseteq \mathrm{FO}(\sigma)$, $\mathfrak{H} = \mathfrak{H}(\Phi)$ satisfies
$\mathfrak{H} \models \alpha \;\Leftrightarrow\; \Phi \vdash \alpha$ for atomic $\alpha \in \mathrm{FO}(\sigma)$

in general, not compatible with $\neg$, $\vee$ or $\exists$ (!)

for Henkin sets $\Phi \subseteq \mathrm{FO}(\sigma)$, $\mathfrak{H} = \mathfrak{H}(\Phi)$ satisfies
$\mathfrak{H} \models \varphi \;\Leftrightarrow\; \Phi \vdash \varphi$ for all $\varphi \in \mathrm{FO}(\sigma)$

Henkin sets $\Phi \subseteq \mathrm{FO}(\sigma)$ characterised by

- maximal consistency:
  for all $\varphi \in \mathrm{FO}(\sigma)$, precisely one of $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$
- provision of witnesses:
  for every $\exists x\varphi \in \mathrm{FO}(\sigma)$ ex. some $t \in T_\sigma$ s.t. $\Phi \vdash \exists x\varphi \rightarrow \varphi\frac{t}{x}$

Henkin sets towards completeness proof

goal: for consistent $\Phi \subseteq \mathrm{FO}(\sigma)$ find Henkin set $\hat{\Phi} \supseteq \Phi$,
if necessary, in extended signature $\hat{\sigma} \supseteq \sigma$

different cases, of different combinatorial status

- countable σ/countable $\mathrm{FO}(\sigma)$:
  maximal consistency through inductive choices

  $\mathrm{Var} \setminus \mathrm{free}(\Phi)$ infinite:
  can inductively use 'fresh' variables as witnesses,
  else: constants or renaming of variables (simple)

- uncountable σ, general case:
  maximal consistency through AC/Zorn

  use new constants as witnesses (chain construction)

the simple case: countable inductive processes

$\Phi \subseteq \mathrm{FO}(\sigma)$, σ and $\mathrm{FO}(\sigma)$ countable, $\mathrm{Var} \setminus \mathrm{free}(\Phi)$ infinite:

(1) from consistent Φ to consistent $\hat{\Phi}$ with witnesses:
inductively can use variables as witnesses

(2) from consistent Φ to maximally consistent $\hat{\Phi}$:
inductively can add either φ or $\neg\varphi$

crucial finiteness property of consistency:

$\mathrm{cons}(\Phi_i)$ for all $i \in \mathbb{N} \;\Rightarrow\; \mathrm{cons}(\bigcup_{i\in\mathbb{N}} \Phi_i)$

in fact, $\mathrm{cons}(\Phi)$ follows from $\mathrm{cons}(\Phi_0)$ for all finite $\Phi_0 \subseteq \Phi$
$\rightsquigarrow$ compactness (later)

the general case: Zorn's lemma

(1) from consistent Φ to consistent $\hat{\Phi}$ with witnesses:
inductively use new constants as witnesses
in countable chain construction

(2) from consistent Φ to maximally consistent $\hat{\Phi}$:
apply Zorn's lemma to find $\hat{\Phi}$ as $\subseteq$-maximal element
among all consistent extensions of Φ

crucial finiteness property of consistency:
inductive nature of the partial ordering of consistent sets