Syntax & Semantics

symbols, signatures	structures
terms, formulae, sentences	their interpretations
	over structures

formal proof	$\vdash$	**consequence**
syntactic derivation		semantic implication
derivability	$\vdash$	**validity**
consistency	$\vdash$	**satisfiability**
$\vdash$		$\models$

towards **Gödel's Completeness Theorem**

Kurt Gödel, 1906–1978

formal proof system: a sequent calculus

sequent: finite string of formulae $\Gamma\ \varphi$

antecendent $\Gamma \subseteq \mathrm{FO}$ finite sequence
(unordered, possibly empty)

succedent $\varphi \in \mathrm{FO}$

semantics of sequent (validity): $\Gamma\ \varphi$ valid if $\Gamma \models \varphi$

sequent calculus: rule-based calculus for the syntactic generation of the *derivable sequents*

soundness (correctness): only valid sequents are derivable

completeness (weak form)**:** all valid sequents are derivable

sequent calculus

sequent calculus rules: $\dfrac{\text{premise sequents}}{\text{conclusion sequent}}$

idea: sequents as proof snapshots;
sequent rules as legitimate proof steps

examples: $\dfrac{\Gamma\ (\varphi \wedge \psi)}{\Gamma\ \varphi}$ $\qquad$ $\dfrac{\Gamma\varphi_1\ \varphi \qquad \Gamma\varphi_2\ \varphi}{\Gamma(\varphi_1 \vee \varphi_2)\ \varphi}$

here: a sequent calculus $\mathcal{S}$
for $\mathrm{FO}(\sigma)$ with $=, \neg, \vee, \exists$
(without $\wedge, \to, \leftrightarrow, \forall$)

sequent calculus rules

types of rules:

- rules for assumption/antecedent (weakening)
- propositional rules for $\neg, \vee$
- quantifier rules for $\exists$
- equality rules for $=$

assumption/antecedent rules

$$\text{(Ass)}\ \frac{}{\Gamma\ \varphi}\quad \text{for } \varphi \in \Gamma \qquad\qquad \text{(Ant)}\ \frac{\Gamma\ \varphi}{\Gamma'\ \varphi}\quad \text{for } \Gamma \subseteq \Gamma'$$

sequent calculus

propositional rules:

$$\text{(}\vee\text{A)}\ \frac{\Gamma\varphi_1\ \varphi \qquad \Gamma\varphi_2\ \varphi}{\Gamma(\varphi_1 \vee \varphi_2)\ \varphi} \qquad \text{(}\vee\text{S)}\ \frac{\Gamma\ \varphi_i}{\Gamma\ (\varphi_1 \vee \varphi_2)}\quad \text{for } i = 1, 2$$

$$\text{(CD)}\ \frac{\Gamma\psi\ \varphi \qquad \Gamma\neg\psi\ \varphi}{\Gamma\ \varphi} \qquad\qquad \text{(Ctr)}\ \frac{\Gamma\neg\varphi\ \psi \qquad \Gamma\neg\varphi\ \neg\psi}{\Gamma\ \varphi}$$

sequent calculus

quantifier rules:

$$(\exists\mathrm{A})\ \frac{\Gamma\, \varphi(y/x)\ \psi}{\Gamma\, \exists x\varphi\ \psi} \qquad y \notin \mathrm{free}(\Gamma, \exists x\varphi, \psi)$$

$$(\exists\mathrm{S})\ \frac{\Gamma\ \varphi(t/x)}{\Gamma\ \exists x\varphi}$$

side condition in (∃A) crucial for correctness!

sequent calculus

equality rules

$$(=)\ \frac{}{\Gamma\ t = t} \qquad (\mathrm{Subst})\ \frac{\Gamma\ \varphi(t/x)}{\Gamma\ t = t'\ \varphi(t'/x)}$$