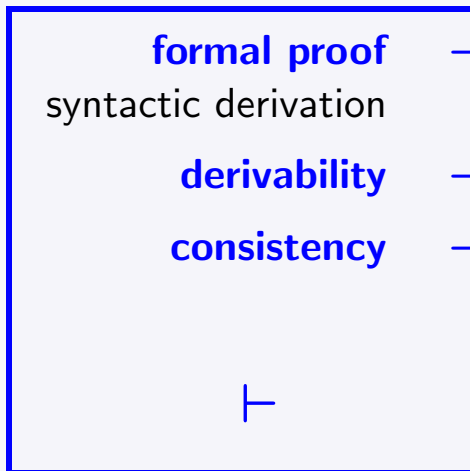


Syntax & Semantics

symbols, signatures
terms, formulae, sentences

structures
their interpretations
over structures



consequence
semantic implication

validity

satisfiability

\models

towards **Gödel's Completeness Theorem**

Kurt Gödel, 1906–1978



formal proof system: a sequent calculus

sequent: finite string of formulae $\Gamma \varphi$

antecedent $\Gamma \subseteq \text{FO}$ finite sequence
(unordered, possibly empty)

succedent $\varphi \in \text{FO}$

semantics of sequent (validity): $\Gamma \varphi$ valid if $\Gamma \models \varphi$

sequent calculus: rule-based calculus for the syntactic generation of the *derivable sequents*

soundness (correctness): only valid sequents are derivable

completeness (weak form): all valid sequents are derivable

sequent calculus

sequent calculus rules:
$$\frac{\text{premise sequents}}{\text{conclusion sequent}}$$

idea: sequents as proof snapshots;
sequent rules as legitimate proof steps

examples:
$$\frac{\Gamma (\varphi \wedge \psi)}{\Gamma \varphi} \quad \frac{\Gamma \varphi_1 \varphi \quad \Gamma \varphi_2 \varphi}{\Gamma (\varphi_1 \vee \varphi_2) \varphi}$$

here: a sequent calculus \mathcal{S}
for $\text{FO}(\sigma)$ with $=, \neg, \vee, \exists$
(without $\wedge, \rightarrow, \leftrightarrow, \forall$)

sequent calculus rules

types of rules:

- rules for assumption/antecedent (weakening)
- propositional rules for \neg, \vee
- quantifier rules for \exists
- equality rules for $=$

assumption/antecedent rules

$$\text{(Ass)} \quad \frac{}{\Gamma \varphi} \quad \text{for } \varphi \in \Gamma \qquad \text{(Ant)} \quad \frac{\Gamma \varphi}{\Gamma' \varphi} \quad \text{for } \Gamma \subseteq \Gamma'$$

sequent calculus

propositional rules:

$$\text{(}\vee\text{A)} \quad \frac{\Gamma \varphi_1 \varphi \quad \Gamma \varphi_2 \varphi}{\Gamma (\varphi_1 \vee \varphi_2) \varphi} \qquad \text{(}\vee\text{S)} \quad \frac{\Gamma \varphi_i}{\Gamma (\varphi_1 \vee \varphi_2)} \quad \text{for } i = 1, 2$$

$$\text{(CD)} \quad \frac{\Gamma \psi \varphi \quad \Gamma \neg \psi \varphi}{\Gamma \varphi} \qquad \text{(Ctr)} \quad \frac{\Gamma \neg \varphi \psi \quad \Gamma \neg \varphi \neg \psi}{\Gamma \varphi}$$

sequent calculus

quantifier rules:

$$\begin{array}{cc} (\exists A) \frac{\Gamma \varphi(y/x) \psi}{\Gamma \exists x \varphi \psi} & (\exists S) \frac{\Gamma \varphi(t/x)}{\Gamma \exists x \varphi} \\ y \notin \text{free}(\Gamma, \exists x \varphi, \psi) & \end{array}$$

side condition in $(\exists A)$ crucial for correctness!

sequent calculus

equality rules

$$\begin{array}{cc} (=) \overline{\Gamma t = t} & (\text{Subst}) \frac{\Gamma \varphi(t/x)}{\Gamma t = t' \varphi(t'/x)} \end{array}$$