semantics & assignments: coincidence lemma

only the assignment to free variables matters

if $\beta \restriction \mathrm{free}(\varphi) = \beta' \restriction \mathrm{free}(\varphi)$

then $(\mathfrak{A}, \beta) \models \varphi$ iff $(\mathfrak{A}, \beta') \models \varphi$

proof by syntactic induction,
based on auxiliary claim for
$t \in T_\sigma$ and $\beta \restriction \mathrm{var}(t) = \beta' \restriction \mathrm{var}(t)$

in particular, semantics of sentences independent of assignment (!)
σ-sentences define classes of σ-structures ($\rightsquigarrow$ elementary classes)

convention: write $\mathfrak{A}, \mathbf{a} \models \varphi$ or $\mathfrak{A} \models \varphi[\mathbf{a}]$
indicating just the assignment $\mathbf{a}$ to variables $\mathbf{x}$ in $\varphi = \varphi(\mathbf{x})$

semantics & isomorphisms: isomorphism lemma

sanity check!

if $(\mathfrak{A}, \beta) \simeq (\mathfrak{A}', \beta')$

then $(\mathfrak{A}, \beta) \models \varphi$ iff $(\mathfrak{A}', \beta') \models \varphi$

proof by syntactic induction,
based on auxiliary claim about
compatibility of isomorphisms $f : \mathfrak{A} \simeq \mathfrak{A}'$
with $\mathfrak{I}(t)/\mathfrak{I}'(t)$ for $\mathfrak{I} = (\mathfrak{A}, \beta)/\mathfrak{I}' = (\mathfrak{A}', f \circ \beta)$

in particular, elementary classes are closed under isomorphism (!)

basic semantic notions

consequence relation $\varphi \models \psi$

$\varphi \models \psi$ if, for all interpretations $\mathfrak{I} = (\mathfrak{A}, \beta)$, $\mathfrak{I} \models \varphi$ implies $\mathfrak{I} \models \psi$

logical equivalence $\varphi \equiv \psi$

$\varphi \equiv \psi$ if, for all interpretations $\mathfrak{I} = (\mathfrak{A}, \beta)$, $\mathfrak{I} \models \varphi$ iff $\mathfrak{I} \models \psi$

validity & satisfiability

$\varphi \in \mathrm{FO}(\sigma)$ satisfiable if $\mathfrak{I} \models \varphi$ for some σ-interpretation $\mathfrak{I}$

$\varphi \in \mathrm{FO}(\sigma)$ valid if $\mathfrak{I} \models \varphi$ for every σ-interpretation $\mathfrak{I}$

elementary and Δ-elementary classes

for sentence $\varphi \in \mathrm{FO}_0(\sigma)$:
$\mathrm{Mod}(\varphi)$ denotes the class (!) of all models of φ
$\mathrm{Mod}(\varphi) = \{\mathfrak{A} : \mathfrak{A} \models \varphi\}$

similarly $\mathrm{Mod}(\Phi) := \bigcap\{\mathrm{Mod}(\varphi) : \varphi \in \Phi\}$
for sets of sentences $\Phi \subseteq \mathrm{FO}_0(\sigma)$

elementary classes

a class $\mathcal{C}$ of σ-structures is an elementary class if $\mathcal{C} = \mathrm{Mod}(\varphi)$ for some sentence $\varphi \in \mathrm{FO}_0(\sigma)$

Δ-elementary classes

a class $\mathcal{C}$ of σ-structures is a Δ-elementary class if $\mathcal{C} = \mathrm{Mod}(\Phi)$ for some set $\Phi \subseteq \mathrm{FO}_0(\sigma)$ of sentences

examples

elementary classes:

groups, rings, fields, . . . , boolean algebras, . . .
graphs, undirected loop-free graphs, . . .
linear orderings, dense linear orderings, partial orderings, . . .
equivalence relations, . . .

Δ-elementary classes:

infinite sets, . . .
fields of characteristic 0, algebraically closed fields, . . .
acyclic graphs, . . .

model-theoretic challenge: how to show that some class is *not* elementary or not even Δ-elementary?

substitution

of terms t for variables x, *where they occur free in* φ *(!)*

for tuple $\mathbf{x} = (x_1, \dots, x_n)$ of pairwise distinct variables
and tuple $\mathbf{t} = (t_1, \dots, t_n)$ of terms $t_i \in T_\sigma$,

the syntactic operation
$$\begin{array}{ccc} \mathrm{FO}(\sigma) & \longrightarrow & \mathrm{FO}(\sigma) \\ \varphi & \longmapsto & \varphi\frac{\mathbf{t}}{\mathbf{x}} \end{array}$$

is such that, for all σ-interpretations $\mathfrak{I} = (\mathfrak{A}, \beta)$,

$$\mathfrak{I} \models \varphi\frac{\mathbf{t}}{\mathbf{x}} \quad \text{iff} \quad \mathfrak{I}\frac{\mathbf{t}^{\mathfrak{I}}}{\mathbf{x}} \models \varphi \qquad \text{(substitution lemma)}$$