Ehrenfeucht–Fraïssé theorem

theorem

for *finite* relational σ and
σ-structures $\mathfrak{A}, \mathfrak{B}$ with parameters $\mathbf{a} \in A^n, \mathbf{b} \in B^n$:

$$\mathfrak{A}, \mathbf{a} \simeq_m \mathfrak{B}, \mathbf{b} \quad \Leftrightarrow \quad \mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$$

remarks:

- useful for inexpressibility proofs:
  find $(\mathfrak{A}_m \in \mathcal{C}, \mathfrak{B}_m \notin \mathcal{C})_{m \in \mathbb{N}}$ such that $\mathfrak{A}_m \simeq_m \mathfrak{B}_m$,
  to show that $\mathcal{C}$ is *not* an elementary class
- unlike compactness, this works to show FO inexpressibility also in restriction to non-elementary classes (e.g., in the class of all finite models, where compactness is unavailable)

the Ehrenfeucht–Fraïssé method

Karp's theorem

for arbitrary relational σ and σ-structures $\mathfrak{A}, \mathfrak{B}$
with parameters $\mathbf{a} \in A^n, \mathbf{b} \in B^n$:

$$\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{part}} \mathfrak{B}, \mathbf{b} \quad \Leftrightarrow \quad \mathfrak{A}, \mathbf{a} \equiv_{\mathrm{FO}_\infty} \mathfrak{B}, \mathbf{b}$$

theorem

for countable $\mathfrak{A}, \mathfrak{B}$:

$$\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{part}} \mathfrak{B}, \mathbf{b} \quad \Leftrightarrow \quad \mathfrak{A}, \mathbf{a} \simeq \mathfrak{B}, \mathbf{b}$$

(obtain isomorphism $\mathfrak{A}, \mathbf{a} \simeq \mathfrak{B}, \mathbf{b}$ as b&f limit)

remark: for finite $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}, \mathbf{a} \simeq \mathfrak{B}, \mathbf{b}$ follows from
$\mathfrak{A}, \mathbf{a}_m \simeq \mathfrak{B}, \mathbf{b}$ for sufficiently large m (why?)

Expressive power of FO: Lindström's thm

Lindström theorem

FO is maximally expressive among logics that satisfy (some very basic closure properties and)

- compactness property:
  Φ unsatisfiable
  $\Rightarrow$ some finite $\Phi_0 \subseteq \Phi$ is unsatisfiable
- Löwenheim–Skolem property:
  Φ countable and satisfiable
  $\Rightarrow$ Φ is satisfiable in countable models

idea: adjoining to FO any φ that is not FO-expressible, in the presence of compactness and Löwenheim–Skolem, would violate invariance under $\simeq$:
construct $\mathfrak{A} \simeq \mathfrak{B}$ with $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \not\models \varphi$

Lindström thm

the game-based proof idea:

for φ that is not expressible in FO find, by compactness,

$\mathfrak{A} \equiv \mathfrak{B}$ with $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$, i.e.,

$\mathfrak{A} \simeq_{\mathrm{fin}} \mathfrak{B}$ with $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$

in FO-encoded environment can boost this, by compactness, to

$\mathfrak{A} \simeq_{\mathrm{part}} \mathfrak{B}$ with $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$

by Löwenheim–Skolem even obtain countable such $\mathfrak{A}, \mathfrak{B}$, so that

$\mathfrak{A} \simeq \mathfrak{B}$ with $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$ (which is absurd)