Expressive power of FO: Ehrenfeucht–Fraïssé method

proviso: purely relational signatures

local/partial isomorphisms in $\mathrm{Part}(\mathfrak{A}, \mathfrak{B})$:
partial maps form $\mathfrak{A}$ to $\mathfrak{B}$ s.t. $p \colon \mathfrak{A}\restriction\mathrm{dom}(p) \simeq \mathfrak{B}\restriction\mathrm{image}(p)$
is an isomorphism (of induced substructures)

NB: $(p \colon \mathbf{a} \mapsto \mathbf{b}) \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A}, \mathbf{a} \equiv_0 \mathfrak{B}, \mathbf{b}$

recall: $\mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$ (m-elementary equivalence) if
$(\mathfrak{A}, \mathbf{a} \models \varphi \;\Leftrightarrow\; \mathfrak{B}, \mathbf{b} \models \varphi$ for all $\varphi \in \mathrm{FO}(\sigma)$ of $\mathrm{qr}(\varphi) \leqslant m)$.

$\mathfrak{A}, \mathbf{a} \equiv \mathfrak{B}, \mathbf{b}$ (elementary equivalence) if
$(\mathfrak{A}, \mathbf{a} \models \varphi \;\Leftrightarrow\; \mathfrak{B}, \mathbf{b} \models \varphi$ for all $\varphi \in \mathrm{FO}(\sigma))$.

- $\equiv$ is the limit of its finite approximations $\equiv_m$
- for finite relational signatures, $\equiv_m$ has finite index

back&forth

$p \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ has b&f extensions in $I \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ if

forth: for all $a \in A$ there is some $p' \in I$,
such that $p \subseteq p'$ and $a \in \mathrm{dom}(p')$;

back: for all $b \in B$ there is some $p' \in I$,
such that $p \subseteq p'$ and $b \in \mathrm{image}(p')$;

b&f systems:

- $I \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ is a b&f system if
every $p \in I$ has b&f extensions in I
- $(I_k)_{k \leqslant m}$ or $(I_k)_{k \in \mathbb{N}}$, where $I_k \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$, are b&f systems if
every $p \in I_{k+1}$ has b&f extensions in I_k

degrees of b&f equivalence

m-isomorphy, $\mathfrak{A}, \mathbf{a} \simeq_m \mathfrak{B}, \mathbf{b}$:

$(p\colon \mathbf{a} \mapsto \mathbf{b}) \in I_m$ for some b&f system $(I_k)_{k \leqslant m}$

finite isomorphy, $\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{fin}} \mathfrak{B}, \mathbf{b}$:

$(p\colon \mathbf{a} \mapsto \mathbf{b}) \in I_k$ for all k in some b&f system $(I_k)_{k \in \mathbb{N}}$

partial isomorphy, $\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{part}} \mathfrak{B}, \mathbf{b}$:

$(p\colon \mathbf{a} \mapsto \mathbf{b}) \in I$ for some b&f system I

intuition: associate b&f systems (Fraïssé)
with winning strategies for second player in
model-theoretic games (Ehrenfeucht)

Ehrenfeucht–Fraïssé theorem

theorem

for *finite* relational σ and
σ-structures $\mathfrak{A}, \mathfrak{B}$ with parameters $\mathbf{a} \in A^n, \mathbf{b} \in B^n$:

$$\mathfrak{A}, \mathbf{a} \simeq_m \mathfrak{B}, \mathbf{b} \quad \Leftrightarrow \quad \mathfrak{A}, \mathbf{a} \equiv_m \mathfrak{B}, \mathbf{b}$$

remarks/observations:

- useful for inexpressibility proofs:
  find $(\mathfrak{A}_m \in \mathcal{C}, \mathfrak{B}_m \notin \mathcal{C})_{m \in \mathbb{N}}$ such that $\mathfrak{A}_m \simeq_m \mathfrak{B}_m$,
  to show that $\mathcal{C}$ is not an elementary class
- similarly $\simeq_{\mathrm{part}}$ corresponds to
  equivalence w.r.t. infinitary logic FO_∞
- for countable $\mathfrak{A}, \mathfrak{B}$: $\mathfrak{A}, \mathbf{a} \simeq_{\mathrm{part}} \mathfrak{B}, \mathbf{b}$ implies $\mathfrak{A}, \mathbf{a} \simeq \mathfrak{B}, \mathbf{b}$