## towards the second incompleteness thm

for recursive $\Phi \subseteq \mathrm{Th}(\mathfrak{N})$ with representations
and a fixed total recursive function that enumerates all valid/derivable sequents $\Gamma\varphi$ with $\Gamma \subseteq \Phi$

$W := \{(n, m)\colon\ m\text{-th sequent yields } \Phi \vdash \varphi_n\}$

recursive, hence repesented w.r.t. $\Phi$ by some $\eta(x, y)$

| | |
|---|---|
| $\mathrm{prov}_\Phi(x) := \exists y\, \eta(x, y)$ | "provability in $\Phi$" |
| $\mathrm{cons}_\Phi := \neg\mathrm{prov}_\Phi(\underline{\ulcorner \neg 0 = 0 \urcorner})$ | "consistency of $\Phi$" |

NB: $\mathrm{prov}_\Phi(x)$ over-approximates provability in an arbitrary $\mathfrak{A} \models \Phi$, but captures the intended meaning over $\mathfrak{N} \models \Phi$

similarly for $\mathrm{cons}_\Phi$



## towards the second incompleteness thm

for $\psi(x) := \neg\mathrm{prov}_\Phi(x)$
and its fixpoint sentence $\varphi$ s.t. $\Phi \vdash \varphi \leftrightarrow \neg\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$ find

- $\mathfrak{N} \models \varphi$ and, by consistency of $\Phi$, $\Phi \nvdash \varphi$
- in sufficiently strong $\Phi$ (like PA), also internally get

  $\Phi \vdash \mathrm{cons}_\Phi \rightarrow \neg\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$

  so that consistency of $\Phi$ implies $\Phi \nvdash \mathrm{cons}_\Phi$

### Gödel's second incompleteness theorem

any sufficiently strong, recursive, consistent axiom system $\Phi$ (like ZFC, PA) cannot prove its own consistency: $\Phi \nvdash \mathrm{cons}_\Phi$

## Löb's axioms for provability (modal style)

(L1) $\Phi \vdash \varphi \quad \Rightarrow \quad \Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$

(L1*) $\Phi \vdash \varphi \quad \Leftrightarrow \quad \Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$

(L2) $\Phi \vdash \Big(\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \wedge \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \to \psi \urcorner})\Big) \to \mathrm{prov}_\Phi(\underline{\ulcorner \psi \urcorner})$

(L3) $\Phi \vdash \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \to \mathrm{prov}_\Phi\Big(\underline{\ulcorner \mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner}) \urcorner}\Big)$

- axiomatic characterisation of a reasonable internal encoding of 'provability from $\Phi$'
- satisfied, e.g., by natural formalisation of provability in PA
- (L1),(L2),(L3) and existence of fixpoint formula $\varphi$ for $\psi(x) := \neg\mathrm{prov}_\Phi(x)$ yield both incompleteness theorems



## Löb's axioms and the incompleteness theorems

### Incompleteness I

assuming (L1) and $\Phi \vdash \varphi \leftrightarrow \neg\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$:

- $\Phi$ consistent $\Rightarrow \Phi \nvdash \varphi$
- $\Phi$ consistent and (L1*) $\Rightarrow \Phi \nvdash \neg\varphi$

### Incompleteness II

assuming (L1),(L2),(L3), $\Phi \vdash \varphi \leftrightarrow \neg\mathrm{prov}_\Phi(\underline{\ulcorner \varphi \urcorner})$,
get for $\mathrm{cons}_\Phi := \neg\mathrm{prov}_\Phi(\underline{\ulcorner \neg 0 = 0 \urcorner})$:

- $\Phi$ consistent $\Rightarrow \Phi \nvdash \mathrm{cons}_\Phi$